would suggest that the solution of this problem must have such a discontinuity. If continuous elements are used, this effect naturally cannot be discovered. A comparison of solutions obtained on the basis of statements (2) and (5) with the same number of elements showed that, by using discontinuous elements, a much faster convergence with respect to the functional is obtained. The dependence of the value of the functional on the number of elements is shown in Fig.4, where the continuous line refers to the use of discontinuous elements, and the broken line to the use of continuous elements.

Note finally that the appearance of a discontinuity in the solution can probably be treated as the start of plastic break-up of the material. Thus, by solving a sequence of problems with increasing loads using scheme (5), we can estimate the instant when break-up starts.

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AN APPROACH TO ADAPTIVE GRID CONSTRUCTION FOR NON-STATIONARY PROBLEMS*

N.A. DAR'IN and V.I. MAZHUKIN

A method of constructing adaptive grids dynamically related to the solution is considered.

Introduction

Considerable attention has been devoted in recent years to the construction of difference grids. According to one opinion, a definitive solution of many numerical problems in mathematical physics depends not so much on better methods for the difference approximation of partial differential equations and algorithms for solving difference equations, as on the proper choice of the working grid. Several global trends exist in research on computation, generation and application of grids. One trend is the consideration of movable grids, in which the number of grid-points is fixed but the accuracy of the computed solution can be increased by optimal siting of the grid-points - "optimal" in the sense of the properties of the solution. The principle of optimal grid-point distribution is the basis of methods for constructing so-called adaptive grids. Experience in the use of such grids in heat-transfer and hydrodynamics problems has shown that they are most efficient when dynamically correlated with the solution /2/. As a matter of fact, methods for adaptive grid construction are at present in a state of such intensive development that it is well-nigh impossible to designate specific methods as explicitly preferable to others.

One of the most important elements in the construction of adaptive grids dynamically correlated with the solution is the choice of those characteristics of the numerical solution to be used as parameters regulating the siting of the grid-points. In moving finite-element methods /3, 4/ one uses for that purpose a system of difference equations whose derivation relies on the Galerkin projection method.

In the solution of multidimensional stationary problems /5, 6/ the most widespread approach is apparently to use a variational principle for adaptive grid generation, in various modifications. The basic idea of this approach is to minimize one or more characteristics

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of the numerical solution. In non-stationary problems, besides the variational principle /7/, there are also various heuristic approaches to adaptive grid generation, in which the controlling parameter is either the norm of the approximation error or the rate of variation of the solution /9, 10/.

In this paper we shall describe a method of adaptive grid generation based on an automatic coordinate transformation induced by the unknown solution. We shall consider one possible domain of application of the method - adaptation of the grid to solutions with large gradients.

1. Differential model.

The basic idea of our approach is to describe the behaviour of the adaptive grid points at the differential level, by means of a partial differential equation (or system of equations) which is an essential part of the overall mathematical formulation of the problem. The coordinates of the grid-points, together with the required values of the grid functions, must be determined by solution of a single differential problem. The differential model is constructed by mapping the initial physical space into a certain working space in which the mathematical description of the phenomenon under investigation is augmented by addition of equations describing the distribution of the grid-point as a function of the (unknown) solution. In particular, the grid-points in physical space may cluster together in domains characterized by large gradients, whose location is not known in advance.

To illustrate the method, we will consider a model problem - the propagation of a heat front in a medium. We select a one-dimensional physical space $\Omega_{x,t}: x_0 \leq x \leq x_R, t$ and a certain working space $\Omega_{q,t}: q_0 \leq q \leq q_R, t$. It is assumed that at each instant of time there exists a non-singular transformation

$$x=x(q,t), \quad x(q_0,t)=x_0, \quad x(q_R,t)=x_R, \tag{1}$$

whose actual form will be described later. Suppose that the Jacobian of this transformation is a non-dimensional differentiable function $\psi(q,t)$, which associates an element dq with the element dx. Note that $\psi(q,t)dq$ may have various physical meanings, e.g., it may characterize the mass of the flow element dq. We introduce functions $W(q,t)$ and $Q(q,t)$, where $W(q,t)$ is the heat flow at a given point q at time t and $Q(q,t)$ a certain flow, whose magnitude differs from zero when the corresponding Euler coordinate $x(q)$ varies with time.

The function $\psi(q,t)$ is related to the flow $Q(q,t)$ by the following conservation law:

$$\int_q^{q+\Delta q} [\psi(q,t+\Delta t)-\psi(q,t)]dq = \int_t^{t+\Delta t} [Q(q,t)-Q(q+\Delta q,t)]dt. \tag{2}$$

The law of conservation of energy for an element Δq, in integral form, is

$$\int_q^{q+\Delta q} [\varepsilon(q,t+\Delta t)\psi(q,t+\Delta t)-\varepsilon(q,t)\psi(q,t)]dq = \int_t^{t+\Delta t} [W(q,t)-W(q+\Delta q,t)]dt + \int_t^{t+\Delta t} [\varepsilon(q,t)Q(q,t)-\varepsilon(q+\Delta q,t)Q(q+\Delta q,t)]dt. \tag{3}$$

Here ε is the internal energy. The second integral on the right characterizes the variation in energy due to the flow Q, as defined by the variation of x. Assuming that the integrands are differentiable and letting $\Delta q, \Delta t \to 0$, we can write both conservation laws (2) and (3) in differential form:

$$\frac{\partial}{\partial t}(\psi\varepsilon) = -\frac{\partial W}{\partial q} - \frac{\partial}{\partial q}(Q\varepsilon), \qquad \varepsilon=\varepsilon(T), \tag{4}$$

$$\partial\psi/\partial t = -\partial Q/\partial q, \tag{5}$$

$$\partial x/\partial q = \psi. \tag{6}$$

In terms of the differential model (4)-(6), one can simultaneously describe the temperature behaviour and variation of the linear dimensions of the elements of the physical space $\Omega_{x,t}$. Thus, the relative elongation of the element dx is described by the function ψ. When one goes over to the grid space $\bar{\Omega}_{q,t}$, difference analogues of Eqs.(5) and (6) are used to construct moving grids in the space $\Omega_{x,t}$.

Using the transformation (6), one can write the expression for the heat flow $W(x,t) = -\lambda(T)dT/dx$ in terms of the variables q,t as $W(q,t) = -(\lambda(T)/\psi)\partial T/\partial q$. The general form of the function Q is arbitrary, and only conditions (1) impose the quite natural conditions $Q(q_0,t) = Q(q_R,t) = 0$. The arbitrary nature of Q may be exploited to various ends; in particular, by specifying Q in the form

$$Q=-\chi_0\frac{\partial\psi}{\partial q}-Q_0\frac{\partial}{\partial q}\left(\psi\left|\frac{\partial T}{\partial q}\right|\right) \tag{7}$$

and subsequently solving the system of Eqs.(4)-(6), one can achieve automatic clustering of grid-points in the physical space $\Omega_{x,t}$ in regions of large gradients, i.e., one can construct an adaptive grid dynamically correlated with the unknown solution. The first term in (7) prevents the grid-points from clustering too closely, while the second guarantees that clustering will indeed occur in regions of Euler space characterized by large gradients of the temperature T. In the simplest case, the quantities χ_0 and Q_0 are arbitrary positive constants, to be selected during the computation process. In more general cases, they may be expressed in terms of the thermo-physical characteristics of the medium or other parameters of the problem.

2. Difference approximation.

The domain of integration $q_0<q<q_R$ in the working space is divided into N cells. The values of ε, T, ψ are specified at the mid-points and x, W, Q on the boundaries of the cells. The difference approximation of the differential Eqs.(4)-(6), according to a completely implicit scheme, is as follows:

$$\frac{(\psi\varepsilon)^{j+1}_{i+1/2}-(\psi\varepsilon)^{j}_{i+1/2}}{\tau}=-\frac{W^{j+1}_{i+1}-W^{j+1}_{i}}{h_{i+1/2}}-\frac{(Q\varepsilon)^{j+1}_{i+1}-(Q\varepsilon)^{j+1}_{i}}{h_{i+1/2}}, \tag{8}$$

$$\frac{\psi^{j+1}_{i+1/2}-\psi^{j}_{i+1/2}}{\tau}=-\frac{Q^{j+1}_{i+1}-Q^{j+1}_{i}}{h_{i+1/2}}, \tag{9}$$

$$\frac{x^{j+1}_{i+1}-x^{j+1}_{i}}{h_{i+1/2}}=\psi^{j+1}_{i+1/2}, \tag{10}$$

$$W^{j+1}_{i}=-\frac{\lambda^{j+1}_{i}}{\psi^{j+1}_{i}}\frac{T^{j+1}_{i+1/2}-T^{j+1}_{i-1/2}}{0.5(h_{i+1/2}+h_{i-1/2})}, \tag{11}$$

$$Q^{j+1}_{i}=-\chi_0\frac{\psi^{j+1}_{i+1/2}-\psi^{j+1}_{i-1/2}}{0.5(h_{i+1/2}+h_{i-1/2})}-Q_0\left(\psi^{j+1}_{i+1/2}\frac{|T^{j+1}_{i+1}-T^{j+1}_{i}|}{h_{i+1/2}}-\psi^{j+1}_{i-1/2}\frac{|T^{j+1}_{i}-T^{j+1}_{i-1}|}{h_{i-1/2}}\right)[0.5(h_{i+1/2}+h_{i-1/2})]^{-1},\quad i=1,2,\dots,N, \tag{12}$$

where j is the index of the time level, and τ, h are the time and spatial step-sizes, respectively. The values of the functions ε, T, ψ at integer-valued points were computed from the formula

$$y_i=\frac{\psi_{i-1/2}y_{i+1/2}+\psi_{i+1/2}y_{i-1/2}}{\psi_{i+1/2}+\psi_{i-1/2}}.$$

This system of difference Eqs.(2)-(10) is non-linear and must be solved by iterative procedures. The difference Eqs.(8), (9), together with (11), (12), constitute a five-point scheme; however, if the time derivatives in (12) are substituted from the previous iteration, system (8)-(12) reduces to a three-point scheme and can be solved by matrix pivotal condensation with iterations governed by the non-linearity /11/.

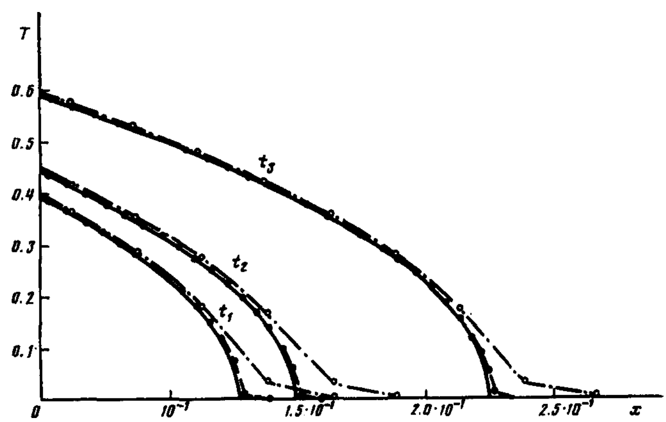

Fig.1

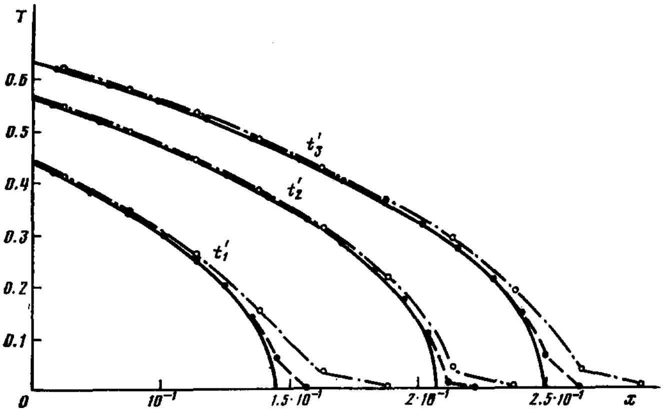

Fig.2

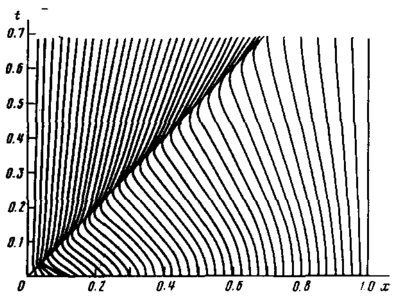

Fig.3

3. Results of calculations.

The possibilities offered by our adaptation method will be illustrated for the case of a one-dimensional non-stationary problem of heat-front propagation. Consider the non-linear heat equation

$$\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left(T^n\frac{\partial T}{\partial x}\right), \tag{13}$$

which has a solution of the travelling-wave type /12/:

$$T(x,t)=\begin{cases}(Dn)^{1/n}(Dt-x)^{1/n}, & x<Dt,\\ 0, & x\geq Dt,\end{cases}$$

where D and n are positive constants.

The non-linear Eq.(13) was solved numerically using the difference schemes (8) and (9). We used the following boundary and initial conditions:

$$T(q_0,t)=(Dn)^{1/n}(Dt)^{1/n},\quad n=2,\quad D=1,\qquad T(q_R,t)=0,$$

$$T(q,0)=0,\quad \psi(q,0)=1,\quad q_0\leq q\leq q_R,\qquad Q(q_0,t)=Q(q_R,t)=0,\quad q_0=0,\quad q_R=1.$$

The analytical solution (shown in Figs.1, 2 by the solid curves) was compared with the numerical solution, computed both with a fixed grid (the dash-dot curves) and with an adaptive grid (the dashed curves). Fig.1 shows the results for the case in which both grids contained the same number of points, $N=40$. The points on the graphs indicate the positions of the grid-points at different times. Comparison of the curves shows that, though the error obtained using a fixed grid is not large, it increases sharply near the front. Use of an adaptive grid ($\chi_0=1$, $Q_0=10$) with the same total number of points $N=40$ increased the accuracy of the solution both deep inside and at the front of the heat wave, thanks to the greater concentration of grid-points in the region of the non-zero solution. In addition, computation with an adaptive grid having only half the previous number of points $N=20$ yielded a more accurate solution than computation with a fixed grid having 40 points (Fig.2).

Fig.3 is an (x,t)-diagram of the motion of the adaptive grid points ($\chi_0=Q_0=1$). The straight line through the origin corresponds to the position of the heat wave front, and the upper part of the diagram characterizes the position of the grid-points below the curve $T(x,t)\neq 0$. According to the (x,t)-diagram the adaptive grid points cluster together in the region of large gradients, a circumstance which in the final analysis increases the accuracy of the numerical solution.

4. Qualitative analysis of the approximation error.

There is a widespread opinion that numerical computations in strongly non-uniform grids are characterized by low accuracy. This is not the case when adaptive grids are used, however, since the changing distribution of the grid-points and the dynamics of the solution are interrelated. To illustrate this, we present a brief qualitative analysis of the approximation error of the difference scheme (8) in q,t variables. To this end we use the apparatus of differential approximation /13/. The first differential approximation for scheme (8) is

$$\frac{\partial}{\partial t}(\psi\varepsilon)+\frac{\partial}{\partial q}(W+Q\varepsilon)=-{}^1/_2\tau\frac{\partial^2}{\partial q\,\partial t}(W+Q\varepsilon)-$$
$$\frac{1}{3!}\left[\frac{h^2}{4}\frac{\partial^3}{\partial q^3}(W+Q\varepsilon)+2\tau^2\frac{\partial^3}{\partial q\,\partial t^2}(W+Q\varepsilon)\right].$$

According to this expression, the order of magnitude of the approximation error of (8) formally - is $O(\tau+h^2)$, while the actual magnitude of the error is determined by the behaviour of the step-sizes τ and h^2 at the corresponding derivatives of the function $W+Q\varepsilon$. Let us analyse the behaviour of the function $W+Q\varepsilon$ and estimate the values of its derivatives in the neighbourhood of the heat wave front.

In the case of a fixed Eulerian grid $Q=0$, even the frist derivative $\partial W/\partial x=-2^{1/2}(t-x)^{-1/2}$ increases without limit near the front, which naturally implies failure of the approximation in that region. As a consequence, the numerical solution involves a large error (the dash-dot curves in Figs.1, 2).

Fig.4 shows the behaviour of the functions W, $Q\varepsilon$, and $W+Q\varepsilon$ when an adaptive grid is used. The position of the front is indicated by the vertical line. We immediately see that $W+Q\varepsilon$ vanishes at the front and its derivatives are small in its neighbourhood. Thus, the mutual relationship between the flow Q, as determined by the positions of the grid-points, and the solution ε makes the approximation error tend to zero in the region of the largest gradients, thus ultimately increasing the accuracy of the numerital solution (the dashed curves in Figs.1, 2).

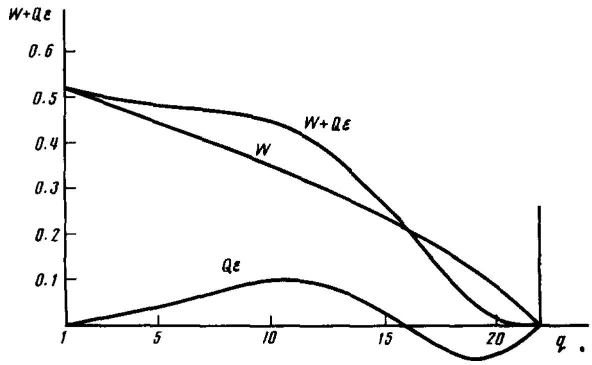

Fig.4

In conclusion, we mention that the method proposed here for adaptive grid generation is easily generalized and can be used to highlight various other properties of the solution, conditioned, e.g., by moving boundaries, and the occurrence and propagation of singular discontinuities and shock waves within the working region.

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