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Инвариантные разностные схемы для параболических уравнений с преобразованием независимых переменных

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Данная работа посвящена построению и исследованию инвариантных разностных схем для нестационарных уравнений при преобразовании независимых переменных. Под инвариантностью разностной схемы понимается ее способность сохранять основные свойсва (устойчивость, аппроксимацию, сходимость и др.) в различных системах координат. Для уравнений параболического типа построены схемы второго порядка аппроксимации на неравномерных сетках, удовлетворяющие свойству инвариантности. Проведено исследование вопросов устойчивости и сходимости соответствующих разностных задач, получены априорные оценки в различных сеточных нормах.

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Present paper concerned with construction and investigation of invariant difference schemes for nonstationary equations during independent variables transformation. Under invariance of difference scheme we understand its ability to preserve basic properties (stability, approximation, convergency, etc.) in various coordinate systems. Difference schemes of the second order of approximation that satisfies to the invariance property are constructed for equations of parabolic type. Stability and convergency investigation of the correspondent difference problems are carried out; a priori estimates in various grid norms are obtained.

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$$y(a) = y(b) = 0,$$

where $\bar{\varphi}_i = f(\bar{x}_i)$, $\bar{x}_i = x_i + (h_{i+1} - h_i)/3$, $\hbar_i = 0{,}5(h_i + h_{i+1})$.

Let $x = x(q)$ be a function that twice continuously differentiable and transforms closed segment $[0,1]$ in $[a,b]$ such that $x(q_i) = x_i$, where q_i is a node of the uniform grid $\omega_{h_q} = \{q_i = i/N, i = \overline{0,N}\}$, and

$$dx/dq = \psi(q) \geq c_0 > 0, \quad x(0) = a. \tag{1.4}$$

After change of variables $x = x(q)$ for the equation (1.1) we get

$$\frac{1}{\psi}\left(\frac{d}{dq}\left(\frac{1}{\psi}\frac{d\bar{u}}{dq}\right)\right) = -\bar{f}(q), \quad 0 < q < 1, \quad \bar{u}(0) = \bar{u}(1) = 0, \tag{1.5}$$

$$\bar{u} = \bar{u}(q) = u(x(q)) = u(x) = u. \tag{1.6}$$

For (1.5) one can write the following difference scheme

$$\frac{1}{\psi_{h,i}}(a\bar{y}_{\bar{q}})_{q,i} = -\bar{f}(\tilde{q}_i), \tag{1.7}$$

where $\tilde{q}_i = x^{-1}(\bar{x}_i)$, $a_i = a(q_i) = \left[\frac{1}{h_q}\int\limits_{q_{i-1}}^{q_i}\psi dq\right]^{-1} = [x_{\bar{q},i}]^{-1}$ is defined from the "best" difference scheme, and following equality holds

$$\bar{f}(\tilde{q}_i) = f(\bar{x}_i).$$

In this case

$$\psi_{h,i} = (x_{i+1/2} - x_{i-1/2})/h_q, \tag{1.8}$$

where $x_{i-1/2} = 0{,}5(x_{i-1} + x_i)$ is a middle of the segment $[x_{i-1}, x_i]$ and $\psi_{hi} - \psi(q_i) = x_{\overset{\circ}{q}} - x'(q_i) = O(h_q^2)$, $x_{\overset{\circ}{q}} = (x_{i+1} - x_{i-1})/2h_q$, whenever the function $x'''(q)$ is restricted when $q \in [0,1]$.

Let us show that the difference schemes (1.3) and (1.7) are invariant by order of approximation. First we demonstrate that the property (1.6) holds for grid solutions of (1.3) and (1.7). It means fulfillment of equalities $y_i = \bar{y}_i$ for any $i = \overline{0,N}$ or that solutions of the difference schemes (1.3) and (1.7) are the same. Let us show that we can obtain the difference scheme (1.3) from (1.7) by algebraic transformations.

Note that the transformation $x = x(q)$ of coordinate system in accordance with (1.4) and condition $x(1) = b$ can be written in form

$$x(q) = a + c_1\int\limits_0^q \psi(q)dq, \qquad c_1 = (b-a)\left(\int\limits_0^1 \psi(q)dq\right)^{-1}.$$

We can assume without loss of generality that $c_1 = 1$. Thus grid transformation is defined as one-to-one correspondence $q_i \mapsto x_i$ of the form

$$x_i = a + \int\limits_0^{q_i}\psi(q)dq, \qquad \psi(q) \geq c > 0. \tag{1.9}$$

Taking into account obvious equalities

$$h_i = x_i - x_{i-1} = \int\limits_0^{q_i}\psi(q)dq - \int\limits_0^{q_{i-1}}\psi(q)dq = \int\limits_{q_{i-1}}^{q_i}\psi(q)dq, \qquad \hbar_i = 0{,}5\int\limits_{q_{i-1}}^{q_{i+1}}\psi(q)dq,$$

for any discrete function y_i we can easily to obtain a sequence of difference expressions

$$(Ly)_i = y_{\bar{x}\hat{x},i} + f(\bar{x}_i) = \frac{1}{\hbar_i}\left(\frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}\right) + f(\bar{x}_i) =$$

$$= \frac{1}{0{,}5\int\limits_{q_{i-1}}^{q_{i+1}} \psi(q)dq} \left(\Big(\frac{1}{h_q} \int\limits_{q_i}^{q_{i+1}} \psi(q)dq \Big)^{-1} \frac{y_{i+1}-y_i}{h_q} - \Big(\frac{1}{h_q} \int\limits_{q_{i-1}}^{q_i} \psi(q)dq \Big)^{-1} \frac{y_i - y_{i-1}}{h_q} \right) + \bar f(\tilde q_i) =$$

$$= \frac{1}{\frac{x_{i+1}-x_{i-1}}{2h_q}} \left(a_{i+1} \frac{y_{i+1}-y_i}{h_q} - a_i \frac{y_i - y_{i-1}}{h_q} \right) + \bar f(\tilde q_i) = \frac{1}{\psi_{h,i}} (a(\psi) y_{\bar q})_q + \bar f(\tilde q_i) = (\bar L y)_i,$$

Here $\tilde q_i$ is a preimage of the point $\bar x_i$ for the correspondence (1.9). Thus any grid function that satisfies to the difference scheme (1.3) is a solution of (1.7) and vice versa.

Characteristic feature of schemes (1.3) and (1.7) ia an equal order of approximation for correspondent differential operators. So, each of them is invariant by order of approximation. In addition, we have to suppose that $x = x(q)$ is sufficiently smooth function.

As shown in (see [20])

$$\delta(x_i) = u_{\bar x \hat x, i} + f(\bar x_i) - \Big(\frac{d^2 u}{dx^2} + f(x) \Big) \Big|_{x=x_i} = O(\hbar_i^2).$$

Taking into account that $\frac{1}{\psi_{h,i}} = (\psi(q))^{-1} \Big|_{q=q_i} + O(h_q^2)$, and $(a(\psi)\bar u_{\bar q})_q = \frac{d}{dq}\Big(\frac{1}{\psi}\frac{d\bar u}{dq}\Big) + O(h_q^2)$ for any function on the grid $\hat\omega_h$ we obtain

$$\bar\delta(q_i) = \frac{1}{\psi_{h,i}} (a(\psi)\bar u_{\bar q})_{q,i} + \bar f(\tilde q_i) - \left(\frac{1}{\psi}\frac{d}{dq}\Big(\frac{1}{\psi}\frac{d\bar u}{dq}\Big) + \bar f(q) \right)\Big|_{q=q_i} = O(h_q^2). \qquad (1.10)$$

Here

$$\bar f(\tilde q_i) = f(x_i + \frac{h_{i+1}-h_i}{3}) = f(x_i) + \frac{h_{i+1}-h_i}{3} f'(x_i) + O(\hbar_i^2) = \bar f(q_i) + \frac{h_q^2 x_{\bar q q}}{\psi^2} \bar f'(q_i) + O(h_q^2) =$$

$$= \bar f(q_i) + O(h_q^2), \qquad h_{i+1} - h_i = x_{i+1} - 2x_i + x_{i-1} = h_q^2 x_{\bar q q}.$$

§2. CONSISTENCY OF GRID NORMS

Usually, stability and convergence investigation of difference scheme occurred in space with some grid norm that is an analogue for correspondent norm in space of functions of continuous argument. Let $u = u(x)$, $a \le x \le b$ be a one-variable function and $\overline{u} = \overline{u}(q)$, $0 \le q \le 1$, where x and q connected by coordinate transformation (1.4). Then norms in spaces of continuous functions C and L_2 are defined in the form $\|u\|_{C[a,b]} = \max\limits_{a\le x\le b} |u(x)|$, $\|\overline{u}\|_{C[0,1]} = \max\limits_{0\le q\le 1} |\overline{u}(q)|$, $\|u\|_{C[a,b]} = \|\overline{u}\|_{C[0,1]}$, $\|u\|_{L_2[a,b]} = \Big\{ \int\limits_a^b u^2(x)dx \Big\}^{1/2} = \Big\{ \int\limits_0^1 \psi \overline{u}^2(q) dq \Big\}^{1/2}$, $\|u\|^2 = (\psi, \overline{u}^2)$. Correspondent seminorms in W_2^1, W_2^2 are:

$$\|u\|_{W_2^1[a,b]} = \Big\{ \int\limits_a^b u'^2(x)dx \Big\}^{1/2} = \Big\{ \int\limits_0^1 \psi^{-1}(q) \overline{u}'^2(q) dq \Big\}^{1/2},$$

$$\|u\|_{W_2^2[a,b]} = \Big\{ \int\limits_a^b u''^2(x)dx \Big\}^{1/2} = \Big\{ \int\limits_0^1 \psi^{-1}(q) (\psi^{-1}(q)\overline{u}'(q))'^2 dq \Big\}^{1/2}.$$

We shall see that similar consistency of norms is occurred in discrete case for the invariant difference scheme (1.7) — (1.8).

Let H_h be a space of grid functions defined on irregular net $\hat{\bar\omega}_h$. As in [5] we define the following norms and scalar products:

$\|v\|_* = (v,v)_*^{1/2} = \Big(\sum\limits_{i=1}^{N-1} \hbar_i v_i^2 \Big)^{1/2}$ — grid norm $L_2(\hat\omega_h)$;

$\|v_{\bar x}]|^2 = (v_{\bar x}, v_{\bar x}] = \sum_{i=1}^{N} h_i v_{\bar x,i}^2$ — grid seminorm $W_2^1(\hat\omega_h^+)$, $\hat\omega_h^+ = \hat\omega_h \cup \{x_N = b\}$;

$\|v_{\bar x\hat x}\|_*^2 = (v_{\bar x\hat x}, v_{\bar x\hat x})_* = \sum_{i=1}^{N-1} \hbar_i v_{\bar x\hat x,i}^2$ — grid seminorm $W_2^2(\hat\omega_h)$;

$\|v\|_C = \max_{x\in\hat\omega_h}|v(x)|$ — uniform norm.

Now suppose that H_{h_q} is a space of grid functions defined on the uniform grid $\overline{\omega}_{h_q}$. For any grid function $\overline{v} \in H_{h_q}$ let us define correspondent grid norms with weight function ψ_h given by (1.8):

$$(\psi_h, \overline{v}^2) = \sum_{i=1}^{N-1} h_q \psi_{hi} \overline{v}_i^2, \quad \left(a, \overline{v}_{\bar q}^2\right] = \sum_{i=1}^{N} h_q a_i \overline{v}_{\bar q,i}^2, \quad \left(\psi_h^{-1}, (a\overline{v}_{\bar q})_q^2\right) = \sum_{i=1}^{N-1} h_q \psi_{hi}^{-1} \left(a_i \overline{v}_{\bar q,i}\right)_{q,i}^2,$$

$$\|\overline{v}\|_C = \max_{q\in\omega_{h_q}} |\overline{v}(q)|,$$

where $a = a(\psi) = [x_{\bar q}]^{-1}$.

It is easy to prove that for invariant difference scheme the consistency of grid norms occurred, i.e.:

$$\|y\|_{C(\hat\omega_h)} = \|\overline{y}\|_{C(\omega_{h_q})}, \quad (\psi_h, \overline{y}^2) = \|y\|_*^2, \tag{2.1}$$

$$\left(a, \overline{y}_{\bar q}^2\right] = \|y_{\bar x}]|^2, \quad \left(\psi_h^{-1}, (a\overline{y}_{\bar q})_q^2\right) = \|y_{\bar x\hat x}\|_*^2. \tag{2.2}$$

Indeed, as a result of invariance and Jacobian approximation (1.8) we get

$$(\psi_h, \overline{y}^2) = \sum_{i=1}^{N-1} (x_{i+1/2} - x_{i-1/2})\, \overline{y}^2(q_i) = \sum_{i=1}^{N-1} \hbar_i y^2(x_i) = \|y\|_*^2.$$

In the same way, we can prove (2.2). For example,

$$\left(a, \overline{y}_{\bar q}^2\right] = \sum_{i=1}^{N} h_q \frac{h_q}{h_i} \frac{(\overline{y}_i - \overline{y}_{i-1})^2}{h_q^2} = \sum_{i=1}^{N} h_i y_{\bar x,i}^2 = \|y_{\bar x}]|^2.$$

§3. A PRIORI ESTIMATES

Let us demonstrate an application of the method of energy inequalities to stability investigation of the difference scheme (1.7) — (1.8) in seminorm (2.2). First we shall prove following statement.

Lemma 1. *For any grid function $\overline{y}(q_i)$ defined on the uniform grid ω_{h_q} such that $\overline{y}(0) = \overline{y}(1) = 0$, it follows that*

$$(\psi_h, \overline{y}^2) \le (l^2/4) \left(a, \overline{y}_{\bar q}^2\right], \quad l = b - a. \tag{3.1}$$

Proof. From (2.1) and embedding [19, p. 37] $\|y\|_* \le (l/2)\|y_{\bar x}]|$ it follows that $(\psi_h, \overline{y}^2) \le (l^2/4)\|y_{\bar x}]|^2$. Our estimate follows from (2.2) and last inequality. The proof is complete.

Now we pay our attention to the problem of stability. Let us consider scalar product of (1.7) onto $\psi_h \overline{y}$. From Green's first difference formula it follows the energy identity:

$$\left(a(\psi_h), \overline{y}_q^2\right] = (\psi_h \overline{\varphi}, \overline{y}). \tag{3.2}$$

Using the Cauchy inequality and embedding (3.1) for scalar product $|(\psi_h\overline{\varphi}, \overline{y})| \le \|\psi_h^{1/2}\overline{\varphi}\|\|\psi_h^{1/2}\overline{y}\| \le (l/2)\|\psi_h^{1/2}\overline{\varphi}\|\|a^{1/2}\overline{y}_{\bar q}]|$ we obtain by means of (3.2) following energy inequality

$$\left(a(\psi_h), \overline{y}_{\bar q}^2\right] \le (l^2/4)\, (\psi_h, \overline{\varphi}^2). \tag{3.3}$$

The last estimate conveys stability of difference scheme by the right side in the grid seminorm $W_2^1(\omega_{h_q})$ with a weight function $a(\psi_h)$.

Note that a priori estimate for solution of difference problem (1.1) can be obtained because of invariance of difference schemes and grid norms in (3.3). In original coordinate system for (1.3) we get $\|y_{\bar{x}}]| \leq (l/2)\|\varphi\|_*$.

By the use of embedding [18, p. 118]

$$\|y\|_C \leq \frac{\sqrt{l}}{2}\|y_{\bar{x}}]|,$$

which holds for any grid function $y(x)$ that defined on arbitrary nonuniform grid $\hat{\omega}_h$ such that $y(0) = y(l) = 0$ we get following estimate of stability

$$\|y\|_C \leq \frac{l^{3/2}}{4}\|\varphi\|_*.$$

§4. DIFFERENCE SCHEMES FOR PARABOLIC EQUATIONS

In rectangular domain $Q_T = \{(x,t):\ a \leq x \leq b,\ 0 \leq t \leq T\}$ let us consider the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \tag{4.1}$$

$$u|_{x=a} = 0, \quad u|_{x=b} = 0, \qquad u(x,0) = u_0(x). \tag{4.2}$$

Assume that the nonuniform grid $\hat{\omega}_{h\tau} = \hat{\omega}_h \times \omega_\tau$ is defined in Q_T. Here $\hat{\omega}_h$ introduced in (1.2) and $\omega_\tau = \{t_j = j\tau,\ j = \overline{0, j_0},\ \tau = T/j_0\}$. Let us suppose that $x = x(q)$ is a change of space variables such that nodes of the grid $\hat{\omega}_h$ turns into nodes of uniform grid $\hat{\omega}_{h_q}$.

After substitution of x, for (4.1) we get

$$\frac{\partial \bar{u}}{\partial t} = \frac{1}{\psi}\frac{\partial}{\partial q}\left(\frac{1}{\psi}\frac{\partial \bar{u}}{\partial q}\right) + \bar{f}(q,t), \tag{4.3}$$

where $\psi = \psi(q)$ is a Jacobian of transformation. It is obvious that

$$\psi(q) = dx/dq.$$

In case of regular change of variables the last formula coincides with (1.4). As it was shown in [20] for the equation (4.1) there is a difference scheme of the second order of approximation in the point $\bar{x}_i$. It has the form

$$y_{(\omega_1,\omega_2)t,i} = y^{(\sigma)}_{\bar{x}\hat{x},i} + \varphi^{(\sigma)}_i, \quad \varphi_i = f(\bar{x}_i, t_j), \quad \sigma = \text{const} > 0, \tag{4.4}$$

where

$$\omega_{1,i} = \frac{h_{i+1} - h_i}{6h_{i+1}} - \left|\frac{h_{i+1} - h_i}{6h_{i+1}}\right|, \quad \omega_{2,i} = \frac{h_i - h_{i+1}}{6h_i} - \left|\frac{h_i - h_{i+1}}{6h_i}\right|, \quad i = \overline{1, N-1},$$

$$u_{(\omega_1,\omega_2),i} = \omega_{1i}u_{i+1} + (1 - \omega_{1i} - \omega_{2i})u_i + \omega_{2i}u_{i-1}, \quad u^{(\sigma)} = \sigma u_i^{j+1} - (1-\sigma)u_i^j.$$

In coordinates (q,t) we can associate to the differential equation (4.3) a difference scheme that can be obtained from the scheme (4.4) by means of algebraic transformations. Taking into account the equality (1.9) we obtain

$$\bar{y}_{(\alpha_1,\alpha_2)t,i} = \frac{1}{\psi_{h,i}}\left(a(\psi)\bar{y}_{\bar{q}}\right)^{(\sigma)}_q - \bar{f}^{(\sigma)}(\bar{q}_i, t), \tag{4.5}$$

$$a_i = \left(\frac{1}{h_q}\int\limits_{q_{i-1}}^{q_i}\psi(q)dq\right)^{-1} = [x_{\bar{q}}]^{-1}, \quad \psi_{h,i} = x_{\overset{\circ}{q},i}, \quad \alpha_1 = \frac{h_q x_{\bar{q}q}}{6x_q} - \left|\frac{h_q x_{\bar{q}q}}{6x_q}\right|, \quad \alpha_2 = -\frac{h_q x_{\bar{q}q}}{6\bar{x}_{\bar{q}}} - \left|\frac{h_q x_{\bar{q}q}}{6\bar{x}_{\bar{q}}}\right|.$$

Let us confirm that the difference scheme (4.5) approximates the equation (4.3) with a second order by q. Note that it is sufficient to prove the equality

$$\bar{u}_{(\alpha_1,\alpha_2)t,i} - \frac{\partial \bar{u}}{\partial t}(q_i, t_j) = O(h_q^2 + \tau),$$

Here we should take into account that for operator by space we have (1.10). Since $\alpha_1 = O(h_q)$, $\alpha_2 = O(h_q)$, we obtain

$$\bar{u}_{(\alpha_1,\alpha_2)t,i} = \alpha_1 \bar{u}_{t,i+1} + (1-\alpha_1-\alpha_2)\bar{u}_{t,i} + \alpha_2 \bar{u}_{t,i-1} = \bar{u}_{t,i} + h_q\alpha_1\bar{u}_{qt,i} - h_q\alpha_2\bar{u}_{\bar{q}t} = \bar{u}_t + O(h_q^2).$$

Further, to prove a priori estimates we shall use the following difference analogues of embedding theorems [20].

Lemma 2. *Let $y(x)$ be a grid function defined on the nonuniform grid $\hat{\bar{\omega}}_h$ and $y(a) = y(b) = 0$. Then the following inequalities holds*

$$\|y\|_* \le \frac{l^2}{4}\|y_{\bar{x}\hat{x}}\|_*, \quad \|y_{\bar{x}}]| \le \frac{l}{2}\|y_{\bar{x}\hat{x}}\|_*, \quad \|y_{\bar{x}}\|_C \le M\|y_{\bar{x}\hat{x}}\|_*, \tag{4.6}$$

where $l = b - a$, $\|y_{\bar{x}}\|_C = \max_{1\le i\le N}|y_{\bar{x},i}|$, $M^2 = \varepsilon + \frac{l}{4} + \frac{l^2(1+c_3)}{8\varepsilon}$, $\varepsilon > 0$ is arbitrary number, and for the constant c_3 we have

$$c_3^{-1} \le \max_i(h_i/h_{i+1}) \le c_3. \tag{4.7}$$

Let us consider the case of implicit scheme (4.4) when $\sigma = 1$. Using the method of energy inequalities and Lemma 2 we shall prove the following theorem.

Theorem 1. *Consider the scheme (4.4) under the assumptions (4.7) and*

$$\tau \ge \frac{c_4}{\varepsilon/2}\max_i|h_{i+1} - h_i|^2, \tag{4.8}$$

where $c_4 = \max_i(\hbar_i/h_i)$, $\varepsilon > 0$ is an arbitrary number. Then the scheme is stable by initial data and right side, and the following estimates holds

$$\begin{aligned} \|y_{\bar{x}\hat{x}}^{n+1}\|_* &\le \mathrm{M}_1\left[\|y_{0\bar{x}\hat{x}}\|_* + \max_{0\le j\le n+1}\|\varphi^j\|_* + \tau\sum_{j=0}^{n}\|\varphi_t^j\|_*^2\right], \\ \|y_{\bar{x}}^{n+1}\|_C &\le \mathrm{M}_2\left[\|y_{0\bar{x}\hat{x}}\|_* + \max_{0\le j\le n+1}\|\varphi^j\|_* + \tau\sum_{j=0}^{n}\|\varphi_t^j\|_*^2\right], \end{aligned} \tag{4.9}$$

where $\mathrm{M}_1 = (1+\tau)^{n+1} \le e^T$, $\mathrm{M}_2 = \mathrm{M}_1\mathrm{M}$ (M defined in Lemma 2).

Proof. Multiplying the equation (4.4) by $-2\tau\hbar_i y_{t\bar{x}\hat{x},i}$ and summing over all internal nodes of the grid $\hat{\omega}_h$, we get

$$2\tau\|y_{t\bar{x}}]|^2 + \tau^2\|y_{t\bar{x}\hat{x}}\|_*^2 + \|\hat{y}_{\bar{x}\hat{x}}\|_*^2 = \|y_{\bar{x}\hat{x}}\|_*^2 + 2\tau(h_+\omega_1 y_{tx} - h\omega_2 y_{t\bar{x}}, y_{t\bar{x}\hat{x}})_* + 2\tau(\hat{\varphi}, y_{t\bar{x}\hat{x}})_*, \tag{4.10}$$

where $h_+ = h_{i+1}$, $h = h_i$, $\hat{\varphi} = \varphi^{j+1}$. Using the Cauchy inequality and the identity $\tau(\hat{v}, y_t)_* = (\hat{v}, \hat{y})_* - (v, y)_* - \tau(v_t, y)_*$ one can see that the following relation holds

$$-\|\hat{y}_{\bar{x}\hat{x}}\|_*^2 + \|y_{\bar{x}\hat{x}}\|_*^2 + 2\tau(\hat{\varphi}, y_{t\bar{x}\hat{x}})_* \le -\|\hat{y}_{\bar{x}\hat{x}} - \hat{\varphi}\|_*^2 + (1+\tau)\|y_{\bar{x}\hat{x}} - \varphi\|_*^2 + \tau(1+\tau)\|\varphi_t\|_*^2.$$

From embedding (4.6), condition (4.7), and the Cauchy inequality, we get the following

$$2\tau(h_+\omega_1 y_{tx}, y_{t\bar{x}\hat{x}})_* \le 2\tau(\max_i|h_+\omega_1|)\|y_{tx}\|_*\|y_{t\bar{x}\hat{x}}\|_* \le 2\tau\sqrt{c_4}(\max_i|h_+\omega_1|)\|y_{t\bar{x}}]|\,\|y_{t\bar{x}\hat{x}}\|_* \le$$

$$\le \frac{2c_4}{\varepsilon}(\max_i|h_+\omega_1|)^2\|y_{t\bar{x}}]|^2 + \frac{\varepsilon\tau^2}{2}\|y_{t\bar{x}\hat{x}}\|_*^2.$$

In the same way,

$$-2\tau(h\omega_2 y_{t\bar{x}}, y_{t\bar{x}\hat{x}})_* \le \frac{2c_4}{\varepsilon}(\max_i |h\omega_2|)^2 \| y_{t\bar{x}}]|^2 + \frac{\varepsilon\tau^2}{2}\|y_{t\bar{x}\hat{x}}\|_*^2.$$

Since $\max_i |h\omega_2| = \max_i |h_+\omega_1| = \max_i |h_+ - h|$ then using the last estimates in (4.10), we have

$$\|\hat{y}_{\bar{x}\hat{x}} - \hat{\varphi}\|_*^2 + 2\left(\tau - \frac{2c_4}{\varepsilon}\max|h_+ - h|^2\right)\|y_{t\bar{x}}]|^2 + (1-\varepsilon)\tau^2\|y_{t\bar{x}\hat{x}}\|_*^2 \le (1+\tau)\|y_{\bar{x}\hat{x}} - \varphi\|_*^2 + \tau(1+\tau)\|\varphi_t\|_*^2$$

Taking into account conditions of the Theorem we obtain

$$\|y_{\bar{x}\hat{x}}^{n+1} - \varphi^{n+1}\|_*^2 \le (1+\tau)^{n+1}\|y_{0\bar{x}\hat{x}} - \varphi_0\|^2 + \tau\sum_{j=0}^{n}(1+\tau)^{n-j+1}\|\varphi_t^j\|_*^2$$

From here and (4.6) the Theorem follows.

Remark. Recall that the scheme (4.4) possesses the second order of approximation. For the solution of the scheme (4.4) as well as for its derivative one can easy to prove, by means of (4.9), that

$$\|z\|_C, \ \|z_{\bar{x}}\|_C \le C_0(h^2 + \tau),$$

where $z = y - u$, $C_0 = \text{const}$, $h = \max_i h_i$.

From invariance of the difference schemes, consistency of the grid norms, and inequalities (4.9) we immediately obtain the corresponding a priori estimates for the solution of the scheme (4.5) defined on the uniform grid $\bar{\omega}_{h_q}$. Moreover, the condition (4.8) on τ can be formulated in terms of functions on the net $\bar{\omega}_{h_q}$ as $\tau \ge c_5 h_q^2$, $c_5 = (2c_4/\varepsilon)\max_i |x_{\bar{q}q,i}|$. Note that in case of uniform grid $h_{i+1} - h_i = h_q^2 x_{\bar{q}q} = 0$, i.e. the restriction on τ eliminates, so the schemes are absolutely stable.

Monotone difference schemes of second order of precision. Let us consider an implicit difference scheme that approximate the equation (4.1) with the second order at the point $\bar{x}_i$. Moreover, we shall suppose fulfillment of the monotone condition and for clearness that for steps of the grid $\hat{\omega}_h$ we have $h_{i+1} \ge h_i$. The monotone condition means that for any three–point equation written in the canonical form

$$L[y_i] = A_i y_{i-1} - C_i y_i + B_i y_{i+1} = -F_i, \quad i = 1,2,...,N-1, \tag{4.11}$$

it follows that the conditions of maximum principle (see [18, p. 45]) are holds.

Theorem (maximum principle). *Suppose that for any* $i = 1,2,...,N-1$ *coefficients of the equation (4.11) satisfies property*

$$|A_i| > 0, \quad |B_i| > 0, \quad D_i = |C_i| - |A_i| - |B_i| > 0, \quad i = 1,2,...,N-1. \tag{4.12}$$

Then from conditions

$$L[y_i] \ge 0, \quad (L[y_i] \le 0)$$

which fulfilled for any $i = 1,2,...,N-1$ *it follows that grid function* y_i *different from constant can not achieve it's largest positive (least negative) value at interior points, i.e. when* $i = 1,2,...,N-1$.

In this assumptions the difference scheme for the equation (4.1) has the form

$$y_t + \frac{h_{i+1} - h_i}{3} y_{tx} = \hat{y}_{x\hat{x}} + \varphi, \quad y_0 = y_N = 0, \quad \varphi = \hat{f}(\bar{x}_i). \tag{4.13}$$

Note that unlike the scheme (4.4) derivative by time is approximated by flux. Due to this circumstance it is possible to use the maximum principle for investigation of the scheme. Although estimates obtained by means of maximum principle are carries out in more severe constraints on time step τ it should be noted that the principle is applicable to problems of arbitrary dimension.

From the maximum principle it immediately follows the following statement [18, p. 47].

Corollary. *Suppose that the properties* (4.12) *are satisfied. Then for the solution of the problem*

$$L[y_i] = -F_i, \quad i = 1,2,...,N-1, \quad y_0 = 0, \quad y_N = 0$$

it follows that

$$\|y\|_C \le \left\|\frac{F}{D}\right\|_C. \tag{4.14}$$

Let us represent the difference scheme (4.13) in the form (4.11). For the coefficients of three–point equation, we get

$$A_i = \frac{\tau}{\hbar h_{i+1}} - \frac{h_{i+1}-h_i}{3h_{i+1}}, \quad B_i = \frac{\tau}{\hbar_i h_i}, \quad C_i = 1 + A_i + B_i,$$
$$F_i = \frac{2h_{i+1}+h_i}{3h_{i+1}} y_i + \frac{h_{i+1}-h_i}{3h_{i+1}} y_{i+1} + \tau\varphi_i. \tag{4.15}$$

It is readily seen that positivity of the coefficients (4.15) take place when

$$\tau > \frac{h_{i+1}^2 - h_i^2}{6}.$$

From (4.14) it follows that at every time moment

$$\|y^{j+1}\|_C \le \max_i \left| \frac{2h_{i+1}+h_i}{3h_{i+1}} y_i^j + \frac{h_{i+1}-h_i}{3h_{i+1}} y_{i+1}^j + \tau\varphi_i^j \right| \le$$
$$\le \max_i \{|y_i^j|, |y_{i+1}^j|\} + \tau \max_i |\varphi_i^j| = \|y^j\|_C + \tau\|\varphi^j\|_C.$$

Summing over all $j = 0, 1, ..., n$, we get

$$\|y^{n+1}\|_C \le \|y^0\|_C + \tau \sum_{j=0}^{n} \|\varphi^j\|_C. \tag{4.16}$$

The approximation error of the scheme (4.13) at the point $\bar{x}_i$ is equal by order to $O(\hbar_i^2 + \tau)$. Using (4.16) we shall get an estimate of the order of convergency for (4.13). For $z = y - u$ we get the problem

$$z_{t,i} + \frac{h_{i+1}-h_i}{3} z_{tx,i} = \hat{z}_{\bar{x}\hat{x},i} + \psi_i.$$

From (4.16) we obtain

$$\|z^{n+1}\|_C \le \tau \sum_{j=0}^{n} \|\psi^j\|_C = O(\hbar_i^2 + \tau).$$

The scheme which invariant to (4.13) has the form

$$\bar{y}_{(\alpha_1,\alpha_2)t,i} = \frac{1}{\psi_{h,i}} \left(a(\psi)\hat{\bar{y}}_{\bar{q}}\right)_q - \hat{\bar{f}}(\bar{q}_i, t), \quad \alpha_{1,i} = h_q^2 x_{\bar{q}q,i}/3x_{q,i}, \quad \alpha_{2,i} = 0.$$

The results obtained above and the principle of consistency for the grid norms allows us to derive estimates of stability and convergency in space of variables $(q,\ t)$.

Difference schemes for problems with moving boundaries. Suppose that the domain $\bar{Q}_T$ is a parallelogram: $\bar{Q}_T = \{(x,t):\ a + vt \le x \le b + vt,\ 0 \le t \le T\}$, $v = \text{const} > 0$, $0 \le a < b$. Conditions on the left and right boundaries of the domain as well as initial data for the equation (4.1) we shall suppose as

$$u\,|_{x=a+vt} = 0, \quad u\,|_{x=b+vt} = 0, \quad u(x,0) = u_0(x).$$

The initial domain $\bar{Q}_T$ transforms to rectangular $\overline{D} = \{(q,t) : 0 \le q \le 1,\ 0 \le t \le T\}$ by change of variables $x(q,t) = \psi q + vt' + a$ with $\psi = b - a$. If $\psi = \partial x/\partial q$ is a metric coefficient and $\partial x/\partial t = v$ is a speed of the system movement which in general should be determined, then for the partial derivatives, we get

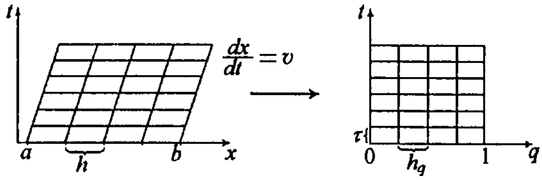
$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \frac{\partial x}{\partial t'}\frac{1}{\psi}\frac{\partial}{\partial q} = \frac{\partial}{\partial t'} - \frac{v}{\psi}\frac{\partial}{\partial q}, \quad \frac{\partial}{\partial x} = \frac{\partial q}{\partial x}\frac{\partial}{\partial q} = \frac{1}{\psi}\frac{\partial}{\partial q}, \quad \frac{\partial^2}{\partial x^2} = \frac{1}{\psi}\frac{\partial}{\partial q}\left(\frac{1}{\psi}\frac{\partial}{\partial q}\right). \tag{4.17}$$

The equation (4.1) can be written in the form

$$\frac{\partial \overline{u}}{\partial t'} - r\frac{\partial \overline{u}}{\partial q} = \frac{\partial}{\partial q}\left(k\frac{\partial \overline{u}}{\partial q}\right) + \overline{f}, \tag{4.18}$$

where $r = v/\psi > 0$, $k = 1/\psi^2 > 0$ are constants.

Note that any uniform grid in calculated space (q,t) generates a uniform grid with the constant step $h = \psi h_q$ in "physical" space (see. the picture): $\overline{\omega}_h^j = \left\{(x_i^j, t_j),\ x_i^j = ih + x_0^j,\ h = \psi h_q,\ x_0^j = a + vt_j,\ i = \overline{0,N}\right\}$, $j = \overline{0, j_0}$.



For the equation (4.18) let us consider a monotone difference scheme of the second order of approximation (see. [18, p. 185])

$$\overline{y}_t = \overline{\varkappa}(k\overline{y}_{\bar{q}}^{(\sigma)})_q + r\overline{y}_q^{(\sigma)} + \overline{\varphi}, \tag{4.19}$$

$$\overline{\varkappa} = 1/(1+R), \quad R = 0,5rh_q/k, \quad \overline{\varphi} = \overline{f}^{(\sigma)}, \quad \overline{y}_0^{j+1} = \overline{y}_N^{j+1} = 0, \quad \overline{y}_i^0 = u_{0i},$$

where $\overline{y}_t = (\overline{y}(q_i, t'_{j+1}) - \overline{y}(q_i, t'_j))/\tau = (\hat{\overline{y}} - \overline{y})/\tau$.

Approximation error. Let us consider the residual $\delta_{h_q} = -\overline{u}_t + v\left(\psi^{-1}\overline{u}_q^{(\sigma)}\right) + +\overline{\varkappa}\psi^{-1}\left(\psi^{-1}\overline{u}_{\bar{q}}^{(\sigma)}\right)_q$ of the scheme (4.19).

Note that $R = O(h_q)$. Using Tailor–series expansion of the function $\overline{u}(q,t')$ at the point's (q_i, t'_j) neighborhood it can be proved that

$$\delta_{h_q} = \tau\sigma\left(\frac{1}{\psi}\frac{\partial}{\partial q}\left(\frac{1}{\psi}\frac{\partial^2 \bar{u}}{\partial t' \partial q}\right) + \frac{1}{\psi}\frac{\partial^2 \bar{u}}{\partial t' \partial q}\right) + O\left((h_q\psi)^2 + \tau\right).$$

Suppose that $\overline{y}$ is a solution of the difference scheme (4.19) on the grid $\omega_{h_q} \times \omega_\tau$ and the function y defined at the node of the moving grid $\overline{\omega}_h^j$. To analyze invariance property let us introduce the following notation: $y = y(x(q_i, t_j), t_j) = \ = \overline{y}(q_i, t'_j) = \overline{y}$, $\hat{y} = y(x(q_i, t_{j+1}), t_{j+1}) = \overline{y}(q_i, t'_{j+1}) = \hat{\overline{y}}$.

From (4.18) and algebraic relations

$$\frac{y_q}{\psi} = \frac{\overline{y}_{i+1}^j - \overline{y}_i^j}{\psi h_q} = \frac{y_{i+1} - y_i}{h} = y_x, \quad \psi^{-1}\left(\psi^{-1}y_{\bar{q}}\right)_q = y_{\bar{x}x}, \quad \frac{\overline{r}}{\overline{k}} = v\psi, \quad R = \frac{\psi h_q v}{2} = \frac{hv}{2}, \tag{4.20}$$

we get the following difference scheme

$$(\hat{y} - y)/\tau - vy_x^{(\sigma)} = \varkappa y_{\bar{x}x}^{(\sigma)} + \varphi, \quad y(x_0^{j+1}, t_{j+1}) = y(x_N^{j+1}, t_{j+1}) = 0, \quad \varkappa = 1/(1 + 0,5hv). \tag{4.21}$$

The last scheme approximates the initial difference equation (4.1) on the grid $\bar{\omega}_h \times \omega_\tau$.

Residual for (4.21) is an implication of (4.20), (4.17), and it has the form

$$\delta_{h_q} = \delta\left(x_i(t_j), t_j\right) = -u_t + vu_x^{(\sigma)} + \varkappa u_{\bar{x}x}^{(\sigma)} = \tau\sigma\left(\frac{\partial^3 u}{\partial t \partial x^2} + \frac{\partial^2 u}{\partial t \partial x}\right) + O\left(h^2 + \tau\right).$$

From here it follows that the local order of approximation in initial physical space remains the same.

To investigate a stability of the scheme (4.21) on the moving grid $\overline{\omega}_h^j$ let us use the general theory of stability [18]. To do that one has to bring a scheme to the canonical form of two–layered operator–difference schemes

$$By_t + Ay = \varphi, \quad y(0) = u_0, \tag{4.22}$$

where $y = (0, y_1^j, y_2^j, ..., y_{N-1}^j, 0)$ is a required vector and A, B are linear operators defined in finite-dimensional space $H = \omega_h^j$.

Suppose that the operators A and B are defined as

$$B = E + \tau\sigma A, \quad A = A_1^+ + A_2, \tag{4.23}$$

$$(A_1^+ y)_i = \begin{cases} 0, & i = 0, \\ -v y_{x,i}, & i = \overline{1, N-1}, \\ 0, & i = N, \end{cases} \tag{4.24}$$

$$(A_2 y)_i = \begin{cases} 0, & i = 0, \\ -\varkappa y_{\bar{x}x}, & i = \overline{1, N-1}, \\ 0, & i = N. \end{cases} \tag{4.25}$$

Let us show positivity of A. Since $A_2 = A_2^* > 0$ it is sufficient to prove that the operator A_1^+ is non-negative. This follow from the relations

$$(A_1^+ y, y) = -\sum_{i=1}^{N-2} hv \left(\frac{y_i + y_{i+1}}{2} \frac{y_{i+1} - y_i}{h} - \frac{h}{2} y_{x,i}^2 \right) + v y_{N-1}^2 = \frac{v}{2} y_1^2 + \frac{hv}{2} \sum_{i=1}^{N-2} h y_{x,i}^2 + \frac{v}{2} y_{N-1}^2 > 0.$$

Further we shall use the following

Lemma 3. *For any function $y(x)$ defined on the grid*

$$\bar{\omega}_h = \{x_i = ih, \quad 0 \le i \le N, \quad x_0 = 0, \quad x_N = l\}$$

and such that $y(0) = y(l) = 0$ the following inequality holds

$$\|y\|_A \le M_1 \|\tilde{A} y\|, \tag{4.26}$$

where $(Ay)_i = -(a y_{\bar{x}})_{x,i}$, $(\tilde{A} y)_i = -(a y_{\bar{x}})_{x,i} - v y_{x,i}$, $M_1 = l c_2 / (2\sqrt{2c_1}(c_2 + hv))$.

Proof. Multiplying $-(v y_x)_i$ by $h y_i$ and summing over all interior nodes of the grid $\bar{\omega}_h$, we get the identity

$$(-v y_x, y) = \frac{hv}{2} \sum_{i=1}^{N-1} h y_{\bar{x},i}^2 + \frac{v}{2} y_{N-1}^2 = \frac{hv}{2} \|y_{\bar{x}}]|^2. \tag{4.27}$$

Since $a \le c_2$, we can see that for $\|y_{\bar{x}}]|^2$ the following inequality holds

$$\|y_{\bar{x}}]|^2 = ((1/a) a y_{\bar{x}}, y_{\bar{x}}] \ge \frac{1}{c_2} \|y\|_A^2.$$

If we combine this with (4.27), we obtain

$$(-v y_x, y) \ge = \frac{hv}{c_2} \|y\|_A^2.$$

On the other hand,

$$\left(\tilde{A} y, y\right) = (-(a y_{\bar{x}})_x - v y_x, y) \ge \left(1 + \frac{hv}{c_2}\right) \|y\|_A^2.$$

Finally, using the Cauchy inequality and the embedding $\|y\| \le \dfrac{2}{2\sqrt{2c_1}} \|y\|_A$, we get

$$(\tilde{A} y, y) \le \frac{l}{2\sqrt{2c_1}} \|\tilde{A} y\| \; \|y\|_A.$$

The last two inequalities proves the Lemma.

Also, to prove a priori estimates we shall use (see. [18, p.345])

Lemma 4. *Let A be a positive and not self-adjoint operator. If $\sigma \ge 0.5$, then for the scheme* (4.22), (4.23) *one has*

$$\|y^{j+1}\| \le \|y_0\| + \|(A^{-1}\varphi)^0\| + \|(A^{-1}\varphi)^j\| + \sum_{k=1}^{j} \tau \|(A^{-1}\varphi)_{\bar{t},k}\|. \tag{4.28}$$

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INTRODUCTION

At present time along with traditional requirements of similarity, conservativity and complete conservativity for computational methods it is also required a fulfillment of adaptivity property. Adaptive methods used in computational practice directed as a rule on a solving of problems in complex irregular domains with moving boundary (including contact and phase one), contact surfaces (shock waves), domains of big gradients, and boundary layers.

There are two main directions of adaptive grids construction – for stationary and nonstationary problems. In most cases of grid construction for multidimensional stationary problems they use a method of coordinate transformation which allows to map a complex geometric domain into rectangle or square with sides of unit length [1]–[3]. An adaptive grid can be obtained by means of functions of inverse mapping [4, 5]. An idea of coordinate transformation also has a lot of applications for adaptive methods for evolutionary problems [6, 7]. However as a result of problem's nonstationary nature such transformation is realized by means of choosing one of moving coordinate systems. Typical samples of such transformations are Lagrange's variables in gas dynamics [8, 9] and coordinate systems connected with front propagation [10]. Very fruitful idea in this direction consists in construction of grids that dynamically adapted to solution [11]—[17]. Using of arbitrary nonstationary coordinate system permit to describe a grid node behaviour by differential model for which that description is a component of mathematical problem. This method proved to be effective for solution of wide range of problems of mathematical physics among which are problems of nonlinear heat conductivity [11, 12], nonlinear transfer (Burger's equation), gas dynamics [13] one- and multidimensional Stephan problems [16, 17].

Differential problems written in different coordinate systems are equivalent from mathematical point of view. It is intrinsic to demand a realization of similar property for difference schemes.

In papers [22, 23] on the base of simplest differential equation was introduced a concept of invariant difference scheme as it's ability to conserve basic properties during discrete transformation of independent variables and also were formulated basic principles of construction of difference schemes.

The purpose of this work is a construction of invariant difference schemes that also conserves a property of the second order of precision by space variable in various coordinate systems. This circumstance has a principal meaning. It is well known that in approximation on ordinary nonuniform grids of the second order derivatives by space as a rule one can achieve only the first order of approximation. In this paper for invariant difference schemes on moving grids a priory estimates of stability and convergency have been obtained. Constriction and investigation of such schemes based on results from [22].

§1. MODEL EQUATION

Definition. We shall say that a difference scheme is called *invariant* by some property if this property conserves in a given class of discrete transformations of independent variables.

For example, if we pass from one coordinate system to another then we can wish to conserve such properties as approximation, stability, convergence, etc.

On the base of elementary second order equation

$$u''(x) = -f(x), \quad a < x < b, \quad u(a) = u(b) = 0 \tag{1.1}$$

we shall demonstrate how to construct and theoretically investigate such schemes. On the segment $[a,b]$ let us introduce an arbitrary nonuniform grid

$$\widehat{\omega}_h = \{x_i = x_{i-1} + h_i,\ i = \overline{1,N},\ x_0 = a,\ x_N = b\}. \tag{1.2}$$

Corresponding difference scheme of the second order of approximation on the grid $\widehat{\omega}_h$ is [20]

$$\frac{1}{\hbar_i}\left(\frac{y_{i+1}-y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}\right) = -\bar{\varphi}_i, \tag{1.3}$$

From here it follows that the difference scheme (4.21) on moving grid is stable in grid norm L_2. More strong estimate one can get by means of Theorem 2 from [21]. Here we cite this theorem as

Lemma 5. *Consider the scheme* (4.22), (4.23) *under the assumptions of Lemma* 4. *Then the scheme is stable by initial data, right side, and the following estimate holds*

$$||Ay^{j+1}|| \le ||Ay^0|| + ||\varphi^0|| + ||\varphi^j|| + \sum_{k=0}^{j-1} \tau ||\varphi_{t,k}||. \tag{4.29}$$

This lemma allows us to get the following a priori estimate for the solution of (4.21)

$$||vy_x^j + y_{\bar{x}x}^j|| \le ||vy_x^0 + y_{\bar{x}x}^0|| + ||\varphi^0|| + ||\varphi^j|| + \sum_{k=0}^{j-1} \tau ||\varphi_{t,k}||.$$

From the last estimate, Lemma 3, and the embedding (see. [20]) $||y||_C \le \sqrt{l}/(2\sqrt{c_1})||y||_A$ one can get the estimate of the scheme (4.21) in the norm C

$$||y||_C \le M \left(||vy_x^0 + y_{\bar{x}x}^0|| + ||\varphi^0|| + ||\varphi^j|| + \sum_{k=0}^{j-1} \tau ||\varphi_{t,k}|| \right), \quad M = l^{3/2}/(4\sqrt{2}c_1).$$

Due to consistency of the grid norms $||y||^2 = \sum_{i=1}^{N-1} h(y_i^j)^2 = \sum_{i=1}^{N-1} h_q \psi(\overline{y}_i^j)^2 = \left(\psi, (\overline{y}_i^j)^2\right)$ one can see that the solution $\overline{y}$ of the scheme (4.19) is stable in the calculated space $\omega_{h_q} \times \omega_\tau$. For example, a priori estimate in the grid norm $L_2(\omega_{h_q})$ has the form $||\psi^{1/2}\overline{y}^{j+1}|| \le ||\psi^{1/2}\overline{y}^0|| + ||\psi^{1/2}(A^{-1}\overline{\varphi})^0|| +$ $+ ||\psi^{1/2}(A^{-1}\overline{\varphi})^j|| + \sum_{k=1}^{j} ||\psi^{1/2}(A^{-1}\overline{\varphi})_{\bar{t},k}||$, where $A = A_1^+ + A_2$ is an operator in finite-dimensional Hilbert space H_{h_q}. According to (4.20), (4.24), (4.25) this operator defined like this

$$(A_1^+\overline{y})_i = \begin{cases} 0, & i = 0, \\ -r\overline{y}_{q,i}, & i = \overline{1, N-1}, \\ 0, & i = N, \end{cases}$$

$$(A_2\overline{y})_i = \begin{cases} 0, & i = 0, \\ -\varkappa k\overline{y}_{\bar{q}q}, & i = \overline{1, N-1}, \\ 0, & i = N. \end{cases}$$

Thus the requirement of absolute stability on the moving grid ω_h^j as well as on the rectangular grid ω_{h_q} is fulfilled whenever $\sigma \ge 0.5$.

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