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NUMERICAL METHODS

Finite-Difference Schemes for the Korteweg–de Vries Equation

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1. INTRODUCTION

Nonlinear wave processes have recently been a subject of increasing interest in various fields of physics (optics, plasma physics, radiophysics, and fluid dynamics). The Korteweg–de Vries equation is often used as a model equation for the analysis of waves of small but finite amplitude in dispersive media; this equation was derived for the first time in [1] by expanding the equations of an ideal incompressible fluid in small parameters. A number of important results pertaining to properties of the Korteweg–de Vries equation and to finding exact solutions were obtained in [2–4]. An explicit three-layer finite-difference scheme [5] with approximation order $O(h^2+\tau^2)$ was the first scheme used for the numerical solution of the Korteweg–de Vries equation. The further development of numerical analysis of the Korteweg–de Vries equation resulted in the appearance of finite-difference schemes with improved stability and accuracy properties [6–11].

In the present paper, we analyze the conservativity and stability of a family of explicit and implicit finite-difference schemes for the Korteweg–de Vries equation from the viewpoint of conservation laws. We use the notion of an L_2-conservative finite-difference scheme implying the validity of a grid analog of the conservation law for the solution [12, 13]. This principle is used in the present paper for constructing new classes of three-layer weighted finite-difference schemes. We obtain *a priori* estimates for the solution of finite-difference problems in the nonlinear case.

2. STATEMENT OF THE PROBLEM

The Korteweg–de Vries equation

$$\partial u/\partial t + u\,\partial u/\partial x + \beta\partial^3 u/\partial x^3 = 0, \qquad \beta = \text{const} > 0, \tag{1}$$

which was derived for the problem on long waves in shallow water [1], is the simplest model equation for the analysis of the evolution of waves of small amplitude in a dispersive medium.

In the rectangular domain $\bar{Q} = \{(x,t) : 0 \le x \le l,\ 0 \le t \le T\}$, we consider the Cauchy problem for Eq. (1) with the spatially periodic conditions

$$u(x,0) = u_0(x), \qquad 0 \le x \le l, \tag{2}$$
$$u(x,t) = u(x+l,t), \qquad t \ge 0. \tag{3}$$

3. THE BASIC PROPERTIES OF THE PROBLEM

When constructing a family of finite-difference schemes effectively approximating a differential problem, one must provide the validity of grid analogs of the conservation laws.

Lemma 1 [14, p. 221]. *Let condition* (3) *be satisfied. Then the nonlinear operator*

$$Lu = L_1u + \beta L_2 u, \qquad L_1 u = u\,\partial u/\partial x, \qquad L_2 u = \partial^3 u/\partial x^3, \tag{4}$$

satisfies the relations

$$(L_k u, u) = 0, \qquad k = 1,2; \qquad (Lu,u) = 0, \qquad (u,v) = \int_0^l u(x,t)v(x,t)\,dx. \tag{5}$$

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When constructing conservative and completely conservative finite-difference schemes [10], it is important to know integral characteristics of the differential problem that do not change in the course of time.

Lemma 2. *For the solution of problem* (1)–(3), *one has the conservation laws*

$$E_1(t) = E_1(0), \qquad E_1(t) = \int_0^l u(x,t)dx, \tag{6}$$

$$E_2(t) = E_2(0), \qquad E_2(t) = \int_0^l u^2(x,t)dx. \tag{7}$$

Proof. Relation (6) is obvious. To derive relation (7), we take the inner product of Eq. (1) by $2u$ and use the periodicity condition (3). We obtain the identity $dE_2(t)/dt = 0$; by integrating it with respect to t, we arrive at the desired assertion.

4. THE MODEL EQUATION

We illustrate the basic points in the construction of finite-difference schemes for the nonlinear Korteweg–de Vries equation on the basis of the linear equation

$$\partial u/\partial t + a\,\partial u/\partial x + \beta\,\partial^3 u/\partial x^3 = 0, \qquad a, \beta = \text{const} > 0, \tag{8}$$

equipped with conditions (2) and (3). We have chosen this equation since it is linear but possesses the basic properties of the original nonlinear problem. For example, just as in Lemma 1, we can readily show that property (5) is valid for the linear analog of the operator Lu: $L_0 u = L_{10}u + \beta L_2 u$, $L_{10}u = a\,\partial u/\partial x$. In other words, L_0 is a skew-symmetric operator: $L_0 = -L_0^*$.

5. TWO-LAYER FINITE-DIFFERENCE SCHEMES

In the domain $\bar{Q}$, we introduce the standard uniform grids

$$\omega_{h\tau} = \omega_h \times \omega_\tau, \qquad \bar{\omega}_h = \{x_i = ih,\ i = 0, \ldots, N,\ hN = l\},$$
$$\bar{\omega}_\tau = \{t_n = n\tau,\ n = 0, \ldots, N_0;\ \tau N_0 = T\} = \omega_\tau \cup \{T\}.$$

As a finite-difference approximation to problem (8), (2), (3), we use the simplest explicit finite-difference scheme

$$y_t + a y_{\mathring{x}} + \beta y_{\bar{x}x\mathring{x}} = 0, \qquad i = 0, \ldots, N, \qquad t \in \omega_\tau, \tag{9}$$

with the periodicity conditions

$$y_{i+N} = y_i, \qquad y(x,0) = u_0(x), \qquad x \in \bar{\omega}_h, \tag{10}$$

on the bottom layer, where $\bar{\omega}_h$ and $\bar{\omega}_\tau$ are the uniform grids with respect to the space and time variables with increments h and τ, respectively.

Note that if condition (10) is satisfied, then we obtain the following approximation to the differential matching conditions:

$$y_0 = y_N, \qquad y_{\mathring{x},0} = y_{\mathring{x},N}, \qquad y_{\bar{x}x,0} = y_{\bar{x}x,N}. \tag{11}$$

Let us show that for the explicit scheme (9), conditions (11) remain valid for $t = t_{n+1}$ as well. Since $y_{\mathring{x},0} = y_{\mathring{x},N}$ and $y_{\bar{x}x\mathring{x},0} = (y_2 - 2y_1 + 2y_{N-1} - y_{N-2})\,/\,(2h^3) = y_{\bar{x}x\mathring{x},N}$, it follows from (9) that $\hat{y}_0 = \hat{y}_N$. Now, setting $\hat{y}_{-2} = \hat{y}_{N-2}$, $\hat{y}_{-1} = \hat{y}_{N-1}$, $\hat{y}_{N+1} = \hat{y}_1$, and $\hat{y}_{N+2} = \hat{y}_2$, we obtain the approximation of the matching condition (11) on the $(n+1)$st layer.

To analyze the properties of the finite-difference scheme (9), we reduce it to the canonical form [14, p. 20]

$$B\,(y_{n+1} - y_n)\,/\tau + A y_n = 0, \qquad y_0 = u_0, \tag{12}$$

where $y_n = (y_0^n, y_1^n, \dots, y_N^n)^{\mathrm{T}} \in H$, $u_0 \in H$, and $H = \Omega_h$ is the space of grid functions defined on the grid $\bar{\omega}_h$ and satisfying the periodicity condition $y_i = y_{i+N}$. Then in the scheme (12), we have

$$B = E, \qquad A : \Omega_h \to \Omega_h, \qquad A = aA_1 + \beta A_2, \tag{13}$$

$$(A_1 y)_i = \begin{cases} (y_1 - y_{N-1})/(2h) & \text{for } i = 0, N, \\ y_{\mathring{x},i} & \text{for } i = 1, \dots, N-1, \end{cases} \tag{14}$$

$$(A_2 y)_i = \begin{cases} y_{\bar{x}x,1}/(2h) - y_{\bar{x}x,N-1}/(2h) & \text{for } i = 0, N, \\ y_{\bar{x}x,2}/(2h) - y_{\bar{x},1}/(2h^2) + (y_0 - y_{N-1})/(2h^3) & \text{for } i = 1, \\ y_{\bar{x}x\mathring{x},i} & \text{for } i = 2, \dots, N-2, \\ (y_1 - y_N)/(2h^3) - y_{\bar{x},N}/(2h^2) - y_{\bar{x}x,N-2}/(2h) & \text{for } i = N-1. \end{cases} \tag{15}$$

We equip the linear space Ω_h with the inner product

$$(u, v)_h = (u, v) = (h/2)u_0 v_0 + \sum_{i=1}^{N-1} h u_i v_i + (h/2) u_N v_N$$

and the norm $\|y\| = \sqrt{(y, y)}$.

To study the properties of the operator A, we need the following assertion, which is a consequence of the well-known formulas of summation by parts and related periodicity conditions.

Lemma 3. *The relation*

$$(u_{\mathring{x}}, v) = -(u, v_{\mathring{x}}) \tag{16}$$

is valid for arbitrary grid functions $u, v \in \Omega_h$.

Using Lemma 3, we can readily show that

$$(v_{\bar{x}x\mathring{x}}, v) = 0 \tag{17}$$

for any grid function $v \in \Omega_h$. This, together with (16), implies that

$$(Ay, y) = 0, \tag{18}$$

i.e., $A = -A^*$ is a skew-symmetric operator. Since the scheme is in divergent form, it follows that the grid analog of the differential conservation law (6) is valid, i.e., the scheme is conservative.

Definition 1. A finite-difference scheme is said to be *L_2-conservative* if the grid analog

$$\|y(t)\|^2 = \|y(0)\|^2, \qquad t \in \omega_\tau,$$

of the integral conservation law (7) valid for the original differential system holds for this scheme.

Let us now investigate the scheme (9) from the viewpoint of L_2-conservativity. To this end, we take the inner product of the operator equation (12) by $2\tau y$. Since $y = y^{(0.5)} - 0.5\tau y_t$ and $y^{(\sigma)} = \sigma y_{n+1} + (1-\sigma) y_n$, it follows from (18) that

$$\|y_{n+1}\|^2 - \tau^2 \|y_t\|^2 = \|y_n\|^2 . \tag{19}$$

Hence we have the energy identity $\|y(t)\|^2 - \sum_{t'=\tau}^{t} \tau \|y_{\bar{t}}(t')\|^2 = \|y(0)\|^2$, $t \in \omega_\tau$. The presence of the negative disbalance implies the invalidity of the corresponding conservation law and shows that the scheme is not necessarily stable in the $L_2(\bar{\omega}_h)$-norm of the energy space $H = \Omega_h$.

By [15], the two-layer finite-difference scheme (9), (10) is said to be stable in L_2 if for any $y_n \in H$, the solution y_{n+1} of problem (9), (10) admits the estimate $\|y_{n+1}\| \le \|y_n\|$, $n = 0, 1, \dots$ Since, by virtue of (19), this inequality fails for any y_n, it follows that the scheme (9), (12) is absolutely unstable in L_2.

To analyze the stability of the scheme (12), one could also use the results of [16]. Indeed, since A and B are constant operators and $B^{-1} = E$ exists, it follows that the scheme (12) is stable in H_{B^*B}

if and only if $BA^* + AB^* \geq \tau A^* A$. Since $A = -A^*$ is a skew-symmetric operator ($B = E$), we find that this inequality is equivalent to the condition $A^*A \leq 0$. Thus, the necessary stability condition in H is not satisfied, and the scheme (9), (12) is absolutely unstable in the weakest L_2-metric.

6. THREE-LAYER FINITE-DIFFERENCE SCHEMES

To approximate Eq. (8), we use the three-layer finite-difference scheme

$$y_{\overset{\circ}{t}} + a y_{\overset{\circ}{x}} + \beta y_{\bar{x}x\overset{\circ}{x}} = 0, \qquad (x,t) \in \omega_{h\tau} \tag{20}$$

with the initial conditions

$$y(x,0) = u_0(x), \qquad y_t(x,0) = \bar{u}_0(x) \tag{21}$$

and the periodicity conditions $y_{i+N} = y_i$. When finding the second initial condition (21), one can use, for example, the differential equation (8) itself with $t = 0$. Now we rewrite the scheme (20), (21) in the operator form

$$y_{\overset{\circ}{t}} + Ay = 0, \qquad t \in \omega_\tau, \qquad y_0 = u_0, \qquad y_1 = \bar{u}_0, \tag{22}$$

where $y = y_n \in \Omega_h$ and A is the skew-symmetric operator given by (13)–(15). Note that the analysis of three-layer finite-difference schemes with a nonself-adjoint operator A was performed in the papers [16–18] and the monographs [14, 19, 20]. Following [19, p. 367], we shall show that this scheme is stable for $\tau\|A\| < 1$ and the energy inequality

$$E(t) = E(0), \qquad t \in \omega_\tau, \tag{23}$$

is valid for it, where $E(t) = \|\hat{y}\|^2 + 2\tau(\hat{y}, Ay) + \|y\|^2 > 0$ and $\hat{y} = y(t+\tau)$. To this end, we rewrite the finite-difference equation (22) in the form $\hat{y} + \tau Ay = \check{y} - \tau Ay$ and estimate the squared norms of the left- and right-hand sides:

$$\|\hat{y}\|^2 + 2\tau(\hat{y}, Ay) + \tau^2\|Ay\|^2 = \|\check{y}\|^2 - 2\tau(Ay, \check{y}) + \tau^2\|Ay\|^2. \tag{24}$$

We add $\|y\|^2$ to both sides and take into account the fact that A is a skew-symmetric operator, i.e., $(Ay, \check{y}) = -(y, A\check{y})$. Then we obtain $E(t) = E(t-\tau) = \cdots = E(0)$. Let us now show that $E(t) > 0$. Indeed, $E \geq \|\hat{y}\|^2 - 2\tau\|\hat{y}\|\,\|Ay\| + \|y\|^2 \geq \|y\|^2 - \tau^2\|Ay\|^2 > 0$ if $\tau\|A\| < 1$. Under the more restrictive condition

$$\tau^2\|A\|^2 \leq 1 - \varepsilon, \qquad 0 < \varepsilon < 1, \tag{25}$$

it follows from (23) and (24) that the scheme (20), (21) is stable in the grid L_2-norm, i.e., the *a priori* estimate $\|y\|^2 \leq \varepsilon^{-1}E(0)$ is valid for any $t \in \omega_\tau$.

Thus, the explicit three-layer scheme (20) substantially differs from the two-layer one (9): it is stable in the space $H = \Omega_h$ under condition (25).

To derive stability conditions convenient for numerical verification, we estimate the norm of the operator $A = aA_1 + \beta A_2$ from above. By virtue of definition (14), we have

$$\|A_1 y\|^2 = \sum_{i=1}^{N-1} h y_{\overset{\circ}{x},i}^2 + \frac{1}{4h}(y_1 - y_{N-1})^2 = \frac{1}{4h^2}\left(\sum_{i=1}^{N-1} h(y_{i+1} - y_{i-1})^2 + h(y_1 - y_{N-1})^2\right)$$
$$\leq \frac{1}{h^2}\left[\frac{1}{2}\left(\sum_{i=1}^{N-1} h y_{i+1}^2 + \sum_{i=1}^{N-1} h y_{i-1}^2\right) + \frac{h}{2}\left(y_1^2 + y_{N-1}^2\right)\right] = \frac{1}{h^2}\|y\|^2,$$

whence $\|A_1\| \leq 1/h$. To estimate the norm of the operator A_2, we consider the quantity

$$\|A_2 y\|^2 = 0.5hl_0^2 + hl_1^2 + \sum_{i=2}^{N-2} h y_{\bar{x}x\overset{\circ}{x},i}^2 + hl_{N-1}^2 + 0.5hl_N^2, \tag{26}$$

where, by (15),

$$l_0 = l_N = 0.5h^{-1}\left(y_{\bar{x}x,1} - y_{\bar{x}x,N-1}\right),$$
$$l_1 = 0.5h^{-1}\left(y_{\bar{x}x,2} - y_{\bar{x},1}/h + (y_0 - y_{N-1})/h^2\right),$$
$$l_{N-1} = -0.5h^{-1}\left(y_{\bar{x}x,N-2} - y_{\bar{x},N}/h + (y_1 - y_N)/h^2\right).$$

Using the algebraic inequality $\left(\sum_{k=1}^{p} a_k\right)^2 \le p\sum_{k=1}^{p} a_k^2$, in (26), we consider the term

$$\sum_{i=2}^{N-2} hy_{\bar{x}x\dot{x}}^2 = \frac{1}{4h^6}\sum_{i=2}^{N-2} h\left(y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}\right)^2 \le \frac{1}{h^6}\sum_{i=2}^{N-2} h\left(y_{i+2}^2 + 4y_{i+1}^2 + 4y_{i-2}^2 + y_{i-2}^2\right).$$

In a similar way, we have

$$0.5h\left(l_0^2 + l_N^2\right) \le h^{-6}\left[h\left(4y_1^2 + y_2^2 + y_{N-2}^2 + 4y_{N-1}^2\right)\right],$$
$$hl_1^2 \le h^{-6}\left[h\left(2y_0^2 + 4y_2^2 + y_3^2 + y_{N-1}^2 + 2y_N^2\right)\right],$$
$$hl_{N-1}^2 \le h^{-6}\left[h\left(2y_0^2 + y_1^2 + 4y_{N-2}^2 + y_{N-3}^2 + 2y_N^2\right)\right].$$

Now, summing the resulting estimates, we obtain the relation $\|A_2\| \le \sqrt{10}/h^3$. By virtue of the triangle inequality, we have

$$\|A\| \le \varphi(h), \qquad \varphi(h) = a/h + \beta\sqrt{10}/h^3. \tag{27}$$

The stability of the explicit three-layer finite-difference scheme (20)–(22) with respect to the initial data has been proved under the condition $\tau\|A\| < 1$. Now, using the estimate (27) for the operator norm, we obtain the equivalent requirement $\tau \le \tau_k$, $\tau_k = h^3/(\beta\sqrt{10} + ah^2)$. In other words, the explicit scheme (20), (21) is conditionally stable. The last inequality is often referred to as the Courant stability condition, and the quantity τ_k is referred to as the Courant number.

Remark 1. Let us show that the solution of the finite-difference scheme (22) satisfies the grid analog

$$E_{2_h}(t) = E_{2_h}(0), \qquad t \in \omega_\tau, \tag{28}$$

of the integral conservation law (7), where $E_{2_h}(t) = (y(t+\tau), y(t))$, $y(t) \in \Omega_h$. Indeed, taking the inner product of Eq. (22) by $2\tau y$ and using the identities $2\tau\left(y_{\mathring{t}}, y\right) = E_{2_h}(t_n) - E_{2_h}(t_{n-1})$ and $2\tau(Ay, y) = 0$, we obtain (28). Since

$$(\hat{y}, y) = \left(\|\hat{y}\|^2 + \|y\|^2 - \tau^2\|y_t\|^2\right)/2, \qquad (\hat{y}, y) = \left(\|\hat{y} + y\|^2 - \tau^2\|y_t\|^2\right)/4,$$

it follows that the expression E_{2_h} is not a grid norm. Therefore, in spite of the fact that the grid conservation laws (23) and (28) are valid for the finite-difference scheme (22), this scheme cannot be called L_2-conservative in the sense of Definition 1.

7. WEIGHTED SCHEMES

For the model equation, we consider the manyparameter family of schemes

$$y_{\mathring{t}} + Ay^{(\sigma_1,\sigma_2)} = 0, \qquad t \in \omega_\tau, \qquad y_0 = u_0, \qquad y_{t,0} = \bar{u}_0, \tag{29}$$

where σ_1 and σ_2 are real parameters; moreover, $y^{(\sigma_1,\sigma_2)} = \sigma_1\hat{y} + (1 - \sigma_1 - \sigma_2)\,y + \sigma_2\check{y}$.

Theorem 1. *Let $A \ne A(t)$ and*

$$\sigma_1 \ge \sigma_2, \qquad \sigma_1 + \sigma_2 \ge 1, \qquad A = -A^*, \qquad A : \Omega_h \to \Omega_h \tag{30}$$

in the scheme (29). *Then the finite-difference scheme is stable, and*

$$\|y_{n+1}\|_1^2 + \sum_{k=1}^{n} 2\tau^2\left(\sigma_1 - \sigma_2\right)\left\|y_{\mathring{t},k}\right\|^2 = \|y_1\|_1^2, \tag{31}$$

where

$$\|y\|_1^2 = 0.5\left(\|y\|^2 + \|\check{y}\|^2\right) + 0.5\tau^2\left(\sigma_1 + \sigma_2 - 1\right)\|y_{\bar{t}}\|^2,$$
$$y_{\mathring{t}} = 0.5\left(y_t + y_{\bar{t}}\right) = \left(y_{n+1} - y_{n-1}\right)/(2\tau), \qquad y_{\bar{t}} = \left(y_n - y_{n-1}\right)/\tau.$$

Proof. Since A is a skew-symmetric operator and $y^{(\sigma_1,\sigma_2)} \in \Omega_h$, it follows by multiplication of Eq. (29) by $2\tau y^{(\sigma_1,\sigma_2)}$ that

$$2\tau\left(y_{\mathring{t}}, y^{(\sigma_1,\sigma_2)}\right) = 0. \qquad (32)$$

Taking into account the identity

$$y^{(\sigma_1,\sigma_2)} = 0.5\,(\hat{y} + \check{y}) + \tau\,(\sigma_1 - \sigma_2)\, y_{\mathring{t}} + 0.5\tau^2\,(\sigma_1 + \sigma_2 - 1)\, y_{\bar{t}t}, \qquad (33)$$

we arrive at the relation $\|y_{n+1}\|_1^2 + 2\tau^2\,(\sigma_1 - \sigma_2)\,\|y_{\mathring{t}}\|^2 = \|y_n\|_1^2$, which implies the desired assertion.

We have thereby proved that the scheme (29) is unconditionally stable (without constraints on the relationship between the increments τ and h) if the weights σ_1 and σ_2 satisfy conditions (30).

Theorem 2. *If* $\sigma_1 = \sigma_2 = 0.25$, *then the scheme* (29) *is* L_2*-conservative and the energy relation*

$$\|0.5\,(y_n + y_{n+1})\| = \|0.5\,(y_0 + y_1)\| \qquad (34)$$

is valid.

Proof. Using the identity $y^{(\sigma_1,\sigma_2)} = y + \tau\,(\sigma_1 - \sigma_2)\, y_{\mathring{t}} + 0.5\tau^2\,(\sigma_1 + \sigma_2)\, y_{\bar{t}t}$, we can reduce the three-layer scheme (29) for $v_n = (y_n + y_{n+1})/2$ to the two-layer scheme

$$v_t + Av^{(0.5)} = 0, \qquad v_0 = 0.5\,(y_0 + y_1)\,. \qquad (35)$$

Taking the inner product of the last equation by $2\tau v^{(0.5)}$ in Ω_h and taking into account the relations $2\tau\left(v_t, v^{(0.5)}\right) = \|\hat{v}\|^2 - \|v\|^2$ and $\left(Av^{(0.5)}, v^{(0.5)}\right) = 0$, we obtain the discrete analog (34) of the integral conservation law (7), valid for the grid solution on the half-integer time layer $y_{n+1/2} = 0.5\,(y_n + y_{n+1})$.

Remark 2. The scheme (29) with arbitrary σ_1 and σ_2 is conservative with respect to the solution $y_{n+1/2}$ as well.

Indeed, summing (29) with respect to $x \in \omega_h$ and taking into account the relation $y \in \Omega_h$, we obtain $((y_n + y_{n+1})/2, 1) = ((y_0 + y_1)/2, 1)$.

8. NONLINEAR SCHEMES

Let us consider the nonlinear Korteweg–de Vries equation (1). We construct and investigate completely conservative (i.e., conservative and L_2-conservative) weighted finite-difference schemes and obtain related *a priori* estimates.

As was mentioned above, a finite-difference approximation must preserve the basic properties of the continuous medium. Therefore, it is natural to require that finite-difference analogs of the conservation laws (6) and (7) be satisfied. For the construction of conservative schemes, Tikhonov and Samarskii suggested the integral-interpolation method.

To construct a conservative scheme for the nonlinear Korteweg–de Vries equation, along with the integral-interpolation method, we use the Steklov averaging of the function u^2:

$$u^2 \sim \frac{1}{u_+ - u}\int\limits_{u}^{u_+} u^2 du = \frac{1}{3}\frac{\left(u^3\right)_x}{u_x} = \frac{1}{3}\left(u_+^2 + uu_+ + u^2\right), \qquad (36)$$

where $u_\pm = u\,(x_{i\pm1}, t)$, $t \in \omega_\tau$. Using (36), we construct the explicit three-layer scheme

$$y_{\mathring{t}} + (1/6)\left(\left(y^3\right)_{\bar{x}}/y_{\bar{x}}\right)_x + \beta y_{\bar{x}x\mathring{x}} = 0 \qquad (37)$$

of the second-order approximation, which is algebraically equivalent to the grid equation [5]

$$y_{\mathring{t}} + \bar{y}y_{\mathring{x}} + \beta y_{\bar{x}x\mathring{x}} = 0, \qquad i = 0, \ldots, N. \qquad (38)$$

The initial conditions and periodicity conditions are precisely approximated: $y(x,0) = u_0(x)$, $x \in \bar{\omega}_h$, and $\hat{y}_{i+N} = \hat{y}_i$; here $\bar{y} = (y_+ + y + y_-)/3$. Note that, by virtue of the divergent form of the finite-difference equation (37), we have

$$\sum_{x\in\bar{\omega}_h} h\,(y_n + y_{n+1})/2 = \sum_{x\in\bar{\omega}_h} h\,(y_0 + y_1)/2$$

for any $n = 0, \ldots, N_0 - 1$, i.e., the finite-difference scheme (38) is conservative. We rewrite it in the operator form $y_{\mathring{t}} + Ay = 0$, where the linear operator A defined as

$$Ay = \bar{y}A_1 y + \beta A_2 y \tag{39}$$

is given by (14) and (15). Let us show that, just as in the case of the differential problem [see (5)], we have

$$(Ay, y) = 0 \tag{40}$$

provided that $y \in \Omega_h$. To this end, it suffices to prove the relation $(\bar{y}A_1 y, y) = 0$. Using the identity $\bar{y}y_{\mathring{x}} = \left(yy_{\mathring{x}} + (y^2)_{\mathring{x}}\right)/3$ and Lemma 3, we obtain

$$(\bar{y}A_1 y, y) = (\bar{y}y_{\mathring{x}}, y) = \left((yy_{\mathring{x}}, y) - (y^2, y_{\mathring{x}})\right)/3 = 0.$$

Consequently, relation (40) holds.

9. IMPLICIT CONSERVATIVE SCHEMES

Let us consider the class of three-layer weighted schemes

$$y_{\mathring{t}} + Ay^{(\sigma,\sigma)} = 0, \tag{41}$$

where, by (39), $Ay^{(\sigma,\sigma)} = \bar{y}^{(\sigma,\sigma)}A_1 y^{(\sigma,\sigma)} + \beta A_2 y^{(\sigma,\sigma)}$. Since $(Av, v) = 0$ for $v = y^{(\sigma,\sigma)}$, we have the following assertion.

Theorem 3. *The finite-difference scheme* (41) *with spatially periodic solutions is conservative and L_2-conservative for $\sigma = 0.5$ and $\sigma = 0.25$, and its solution satisfies the following grid analogs of the differential conservation laws* (6) *and* (7) *for any τ, h, and σ:*

$$E_{1_h}(t) = 0, \qquad E_{1_h}(t) = 0.5\,(\hat{y} + y, 1)\,, \qquad t \in \omega_\tau, \tag{42}$$

$$E_1(t) = E_1(0), \qquad E_1(t) = 0.5\int_0^l (\hat{u} + u)\,dx, \qquad t \in [0, T-\tau), \tag{43}$$

$$E_{2_h}^{(\sigma)}(t) = E_{2_h}^{(\sigma)}(0), \qquad E_{2_h}^{(\sigma)} = 0.5\left(\|\hat{y}\|^2 + \|y\|^2\right) + \tau^2(\sigma - 0.5)\,\|y_t\|^2, \tag{44}$$

$$E_2^{(0.5)}(t) = E_2^{(0.5)}(0), \qquad E_2^{(0.5)}(t) = 0.5\left(\int_0^l \hat{u}^2 dx + \int_0^l u^2 dx\right), \tag{45}$$

$$E_{3_h}^{(\sigma)}(t) = E_{3_h}^{(\sigma)}(0), \qquad E_{3_h}^{(\sigma)} = \|y_{n+1/2}\|^2 + \tau^2(\sigma - 0.25)\,\|y_t\|^2, \quad y_{n+1/2} = 0.5\,(\hat{y} + y)\,, \tag{46}$$

$$E_3^{(0.25)}(t) = E_3^{(0.25)}(0), \qquad E_3^{(0.25)} = \int_0^l u^2(x, t + \tau/2)dx. \tag{47}$$

Proof. Using summation over all nodes of the grid for the scheme (41), we can derive identity (42) (that is the conservativity property). [The scheme (41), by (37), can be reduced to the divergent form

$$0.5\,(\hat{y} + y)_{\bar{t}} + (1/6)\left((v^3)_{\bar{x}}/v_{\bar{x}}\right)_x + \beta v_{\bar{x}x\mathring{x}} = 0$$

for $v = y^{(\sigma,\sigma)}$.]

Note that the expressions (43), (45), and (47) are algebraic corollaries to the differential conservation laws (6) and (7). To prove the energy relations (44) and (46), one takes the inner product of the finite-difference scheme (41) by $2\tau y^{(\sigma,\sigma)}$ in H and uses the identities

$$y^{(\sigma,\sigma)} = 0.5\,(\hat{y} + \check{y}) + (\sigma - 0.5)\tau^2 y_{\bar{t}t}, \qquad \tau\,(y_{\mathring{t}}, \hat{y} + \check{y}) = 0.5\tau\left(\|\hat{y}\|^2 + \|y\|^2\right)_{\bar{t}},$$

$$2\tau\,(y_{\mathring{t}}, (\sigma - 0.5)\tau^2 y_{\bar{t}t}) = \tau^3(\sigma - 0.5)\left(\|y_t\|^2\right)_{\bar{t}}, \qquad E_{2_h}^{(\sigma)} = E_{3_h}^{(\sigma)}.$$

By virtue of Definition 1, the L_2-conservativity of the scheme (41) with $\sigma = 0.5$ and $\sigma = 0.25$ in the sense of the approximations (44) and (46) follows from the corresponding differential conservation laws (45) and (47).

Using the idea of the representation of convective terms in the divergent and nondivergent form, we can construct the class of L_2-conservative finite-difference schemes of the form

$$y_{\mathring{t}} + (1/3)\left(y y_{\mathring{x}}^{(\sigma_1,\sigma_2)} + \left(y y^{(\sigma_1,\sigma_2)}\right)_{\mathring{x}}\right) + \beta y_{\bar{x}x\mathring{x}}^{(\sigma_1,\sigma_2)} = 0, \tag{48}$$

which are already not nonlinear with respect to $\hat{y}$. From (16) and (17), we have

$$\left(y y_{\mathring{x}}^{(\sigma_1,\sigma_2)} + \left(y y^{(\sigma_1,\sigma_2)}\right)_{\mathring{x}}, y^{(\sigma_1,\sigma_2)}\right) = 0.$$

Consequently, by Theorem 1, if $\sigma_1 \geq \sigma_2$ and $\sigma_1 + \sigma_2 \geq 1$, then the solution of the finite-difference problem (48), (21) admits the *a priori* estimate (31). By Theorem 3, the scheme (48) is L_2-conservative with respect to the functional $E_{2_h}^{(0.5)}$ for $\sigma_1 = \sigma_2 = 0.5$ and with respect to the functional $E_{3_h}^{(0.25)}$ for $\sigma_1 = \sigma_2 = 0.25$.

Remark 3. The above-constructed scheme (48) is not conservative. However, it can be viewed as the linearization $\overset{k+1}{v}_{\mathring{t}} + (1/3)\left(\overset{k}{v}\overset{k+1}{v}_{\mathring{x}} + \left(\overset{k}{v}\overset{k+1}{v}\right)_{\mathring{x}}\right) + \beta\,\overset{k+1}{v}_{\bar{x}x\mathring{x}} = 0$ of the nonlinear scheme

$$y_{\mathring{t}} + \bar{y}^{(\sigma_1,\sigma_2)} y_{\mathring{x}}^{(\sigma_1,\sigma_2)} + \beta y_{\bar{x}x\mathring{x}}^{(\sigma_1,\sigma_2)} = 0, \tag{49}$$

where $\overset{k+1}{y}_{\mathring{t}} = \left(\overset{k+1}{y} - \check{y}\right)/(2\tau)$, $v = y^{(\sigma_1,\sigma_2)}$, and $\overset{k+1}{v}_{\mathring{x}} = \left(\overset{k+1}{v}_{+} - \overset{k+1}{v}_{-}\right)/(2h)$.

If the iterative process is convergent ($\lim_{k\to\infty} \overset{k}{y} = \hat{y}$), then it provides a solution of the conservative and L_2-conservative scheme (49) for $\sigma_1 = \sigma_2 = 0.5$ or $\sigma_1 = \sigma_2 = 0.25$.

Remark 4. Let us present an L_2-conservative scheme for the parabolic equation

$$\partial u/\partial t = \partial/\partial x(k\,\partial u/\partial x), \qquad k \geq k_0 > 0, \qquad u(x,0) = u_0(x), \qquad u(0,t) = u(l,t) = 0. \tag{50}$$

Let

$$E(t) = \int_0^l u^2(x,t)dx + 2\int_0^t\int_0^l k\left(\frac{\partial u}{\partial x}\right)^2 dx\,dt. \tag{51}$$

Taking the inner product of Eq. (50) by $2u$, we obtain the integral conservation law

$$E(t) = E(0). \tag{52}$$

We can readily see that for the conservative weighted finite-difference scheme $y_t = \left(a y_{\bar{x}}^{(\sigma)}\right)_x$, $y_0 = y_N = 0$, the grid analog of (52) is valid for $\sigma = 0.5$.

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