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# Difference Schemes on Irregular Grids for Equations of Mathematical Physics with Variable Coefficients

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**Abstract**—New difference schemes of second-order approximation on irregular grids with the use of conventional stencils for stationary and nonstationary problems are proposed. For the stencils considered in this paper, the maximum principle holds without constraints on the relation of the coefficients and steps of the space grid (unconditional monotonicity for stationary equations). For multidimensional problems, efficient schemes of vector additive type are constructed. Numerical experiments carried out in this work indicate improved accuracy of new algorithms on coarse grids as compared to the known second-order accuracy schemes of the first approximation order.

## INTRODUCTION

The improvement of the accuracy of a method without extending the conventional stencil of a difference scheme has always been a topical problem in mathematical physics. In the case of a regular grid, numerous examples of the construction of such computational methods of the fourth-order space approximation can be found in [1] for one-dimensional and multidimensional equations of mathematical physics. As a rule, the order of a local approximation error reduces when passing from a regular grid to an irregular one. It was shown in [2–6] that the accuracy of a method can be improved by approximating the original differential equation at certain intermediate points rather than at grid ones. Unfortunately, the results obtained in these works generally cannot be extended to differential equations with variable coefficients.

In the present study, new algorithms of an increased approximation order on irregular grids with the use of standard stencils are proposed for solving stationary and nonstationary problems with variable (including nonlinear) coefficients. The obvious advantages of the proposed difference schemes are the relative simplicity of the computer algorithms implementing them, unconditional monotonicity in the stationary case (validity of the maximum principle without constraints on the relation between the space steps and the variable coefficients), and reduction to the known conservative difference scheme in the case of a regular grid.

Unlike the work [7], we suggest here efficient stable difference schemes of the second-order local approximation on irregular rectangular grids for a parabolic equation of an arbitrary dimension with variable coefficients. Using the maximum principle for derivatives [8, 9], we prove the stability of the solution obtained by the new computational methods of vector additive type in the norm $C$. The numerical experiments carried out in our study reveal another advantage (the main one from the computational viewpoint) of the algorithms investigated here. These algorithms are considerably more accurate than the schemes of the first-order approximation (the order of accuracy can be the same) precisely on the coarse space-time grids. This reduces the computation time required to solve the problem with the prescribed accuracy.

In our study, we obtained the corresponding a priori estimates to the difference solution expressing the stability with respect to the initial data and right-hand side. We proved the second-order convergence of the proposed algorithms to the solution of the original differential problem. The theoretical studies are based on the general theory of the operator difference schemes [1, 10].

## 1. THE ONE-DIMENSIONAL PROBLEM

Consider the simplest differential problem

$$Lu = (k(x)u')' - d(x)u = -f(x), \quad 0 < x < l, \tag{1.1}$$

$$u(0) = \mu_1, \quad u(l) = \mu_2, \quad 0 < k_1 \le k(x) \le k_2, \quad d(x) \ge d_1 > 0. \tag{1.2}$$

When constructing a monotonic scheme of the second-order accuracy on the three-point stencil using an arbitrary irregular grid $\hat{\bar{\omega}}_h = \hat{\omega}_h \cup \gamma_h$, $\hat{\omega}_h = \{x_i = x_{i-1} + h_i, i = 1, 2, \ldots, N-1\}$, $\gamma_h = \{x_0 = 0, x_N = l\}$, we will base on the relationship [2, 3]

$$u_{\bar{x}\hat{x}, i} - u''(\bar{x}_i) = O(\hbar^2), \quad \bar{x}_i = x_i - \tilde{h}_i, \quad \tilde{h}_i = (h_{i+1} - h_i)/3. \tag{1.3}$$

In this case, we use the conventional notation [1] $u_{\bar{x}\hat{x}} = (u_x - u_{\bar{x}})/\hbar$, $u_x = (u_+ - u)/h_+$, $u_{\bar{x}} = (u - u_-)/h$, $\hbar = 0.5(h + h_+)$, $h = h_i$, $h_+ = h_{i+1}$, $v_\pm = v_{i\pm1} = v(x_{i\pm1})$, $v = v_i$. Equation (1.3) implies that $u_{\bar{x}\hat{x}}$ approximates the second derivative to the second order relative to the nongrid point $\bar{x}_i$ (in the case of a regular grid, $\bar{x}_i \equiv x_i$). To construct similar methods for Eq. (1.1), we will use the identity $(ku')' = 0.5((ku)'' + ku'' - k''u)$. Let us replace the differential operator $L$ by the difference operator $L_h$ on the grid $\hat{\omega}_h$:

$$L_h u = 0.5[(ku)_{\bar{x}\hat{x}} + k_{(\beta_1, \beta_2)} u_{\bar{x}\hat{x}} - k_{\bar{x}\hat{x}} u_{(\beta_3, \beta_4)}] - \bar{d} u_{(\beta_5, \beta_6)}, \tag{1.4}$$

where $\bar{d} = d(\bar{x})$, $v_{(\beta_k, \beta_{k+1})} = \beta_{ki} v_{i+1} + (1 - \beta_{ki} - \beta_{k+1,i}) v_i + \beta_{k+1,i} v_{i-1}$, and the spatially varying weighting factors are determined as follows:

$$\begin{gathered}\beta_1 = 0.5(|\tilde{h}| + \tilde{h})/h_+, \quad \beta_2 = 0.5(|\tilde{h}| - \tilde{h})/h, \quad \beta_3 = 0.5(\tilde{h}k_{\bar{x}\hat{x}} - |\tilde{h}k_{\bar{x}\hat{x}}|)/(h_+ k_{\bar{x}\hat{x}}), \\ \beta_4 = -0.5(\tilde{h}k_{\bar{x}\hat{x}} + |\tilde{h}k_{\bar{x}\hat{x}}|)/(hk_{\bar{x}\hat{x}}), \quad \beta_5 = 0.5(\tilde{h} - |\tilde{h}|)/h_+, \quad \beta_6 = -0.5(|\tilde{h}| + \tilde{h})/h.\end{gathered} \tag{1.5}$$

From Eqs. (1.3), (1.5), we have

$$(ku)_{\bar{x}\hat{x}} - (ku)''(\bar{x}) = O(\hbar^2), \quad v_{(\beta_k, \beta_{k+1})} - v(\bar{x}) = O(\hbar^2), \quad k = 1, 3, 5. \tag{1.6}$$

Therefore, the difference scheme

$$L_h y = -\varphi, \quad \varphi = f(\bar{x}), \quad y_0 = \mu_1, \quad y_N = \mu_2 \tag{1.7}$$

approximates the differential problem (1.1), (1.2) to the second order on an arbitrary irregular grid. The obvious advantage of scheme (1.7) is that, in the case of a regular grid ($h_+ = h$), it is reduced to the well-known conservative scheme (see [1]) $(ay_{\bar{x}})_{x,i} - d_i y_i = -f(x_i)$, $a_i = 0.5(k_{i-1} + k_i)$, $y_0 = \mu_1$, $y_N = \mu_2$. Scheme (1.7) can be written in the canonical form (see [1])

$$A_i y_{i-1} - C_i y_i + B_i y_{i+1} = -F_i, \quad i = 1, 2, \ldots, N-1, \quad y_0 = \mu_1, \quad y_N = \mu_2 \tag{1.8}$$

with the coefficients

$$\begin{gathered}A_i = 0.5[(k_{(\beta_1, \beta_2)} + k_{i-1})/(\hbar_i h_i) - \beta_{4i} k_{\bar{x}\hat{x}, i}] - \beta_{6i} \bar{d}_i, \\ B_i = 0.5[(k_{(\beta_1, \beta_2)} + k_{i+1})/(\hbar_i h_{i+1}) - \beta_{3i} k_{\bar{x}\hat{x}, i}] - \beta_{5i} \bar{d}_i, \quad C_i = \bar{d}_i + A_i + B_i, \quad F_i = f(\bar{x}_i).\end{gathered} \tag{1.9}$$

Below, we will use the grid norms

$$\|\cdot\|_C = \max_{1 \le i \le N-1} |\cdot|, \quad \|\cdot\|_{\bar{C}} = \max_{0 \le i \le N} |\cdot|, \quad \|\cdot\|_{C_\gamma} = \max_{i = \overline{0, N}} |\cdot|.$$

To analyze the stability with respect to the boundary conditions and the right-hand side simultaneously (in particular, in the time-dependent case), we use a more convenient estimate proved by the following lemma.

**Lemma 1.** *Suppose that the conditions*

$$A_i > 0, \quad B_i > 0, \quad D_i = C_i - A_i - B_i > 0, \quad i = 1, 2, \ldots, N-1 \tag{1.10}$$

*are fulfilled. Then, the solution to problem* (1.8) *satisfies the estimate*

$$\|y\|_{\bar{C}} \le \max\{\|y\|_{C_\gamma}, \|F/D\|_C\}. \tag{1.11}$$

Conditions (1.10) for coefficients (1.9) can be satisfied by choosing the space weighting factors in the form (1.5), which are adapted to the grid and coefficients of the problem. For example, let us write out the weighting factors $\beta_1$, $\beta_2$, $\beta_3$, and $\beta_4$ in detail (the trivial cases $\tilde{h} = 0$ and $k_{\bar{x}\hat{x}} = 0$ are not considered):

$$\beta_1 = \begin{cases} \tilde{h}/h_+, & \tilde{h} > 0, \\ 0, & \tilde{h} < 0, \end{cases} \qquad \beta_2 = \begin{cases} 0, & \tilde{h} > 0, \\ -\tilde{h}/h, & \tilde{h} < 0, \end{cases}$$

$$\beta_3 = \begin{cases} 0, & \tilde{h}k_{\bar{x}\hat{x}} > 0, \\ \tilde{h}/h_+, & \tilde{h}k_{\bar{x}\hat{x}} < 0, \end{cases} \qquad \beta_4 = \begin{cases} -\tilde{h}/h, & \tilde{h}k_{\bar{x}\hat{x}} > 0, \\ 0, & \tilde{h}k_{\bar{x}\hat{x}} < 0. \end{cases}$$

Expressions $k_{(\beta_1, \beta_2)}$, $-\beta_3 k_{\bar{x}\hat{x}}$ for $\tilde{h} > 0$, and $k_{\bar{x}\hat{x}} < 0$ are written as

$$k_{(\beta_1, \beta_2)} = \frac{\tilde{h}}{h_+}k_+ + \left(1 - \frac{\tilde{h}}{h_+}\right)k = \frac{\tilde{h}}{h_+}k_+ + \frac{2h_+ + h}{3h_+}k > 0,$$

$$-\beta_3 k_{\bar{x}\hat{x}} = -\frac{\tilde{h}}{h_+}k_{\bar{x}\hat{x}} > 0.$$

All other cases are considered in a similar manner.

Thus, for an arbitrary refinement of the irregular grid, coefficients (1.9) of the difference scheme (1.7) satisfy conditions (1.10) (unconditional monotonicity). Hence, the difference scheme (1.7) is stable with respect to the right-hand side and the boundary conditions, and the a priory estimate (1.11) holds true for the solution to this problem. Substituting $y = z + u$ into Eq. (1.7), we obtain the following problem for the error of the method:

$$L_h z = -\psi, \quad \psi = (L_h u + \varphi) - (Lu + f)(\bar{x}), \quad z|_{\gamma_h} = 0. \tag{1.12}$$

According to Eqs. (1.4), (1.6), the approximation error $\psi = O(h^2)$ ($h = \max\limits_{1 \le i \le N} h_i$). Since all the conditions (1.10) of the maximum principle hold true for problem (1.12), we find from estimate (1.11) that $\|z\|_{\bar{C}} \le \|\psi\|_C \le ch^2$; i.e., difference scheme (1.7) provides the second-order convergence to the exact solution.

## 2. DIFFERENCE SCHEMES FOR THE PARABOLIC EQUATION

Within the rectangle $\bar{D} = \{0 \le x \le l, 0 \le t \le T\}$, we seek a continuous solution to the problem

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) + f(x, t), \quad 0 < x < l, \quad 0 < t \le T,$$

$$u(x, 0) = u_0(x), \quad u(0, t) = \mu_1(t), \quad u(l, t) = \mu_2(t).$$

In addition to an arbitrary irregular space grid $\hat{\bar{\omega}}_h$ introduced above, we will consider the regular time grid $\bar{\omega}_\tau = \{t_n = n\tau, n = 0, 1, \ldots, N_0, \tau N_0 = T\} = \omega_\tau \cup \{T\}$. Since the point $\bar{x}_i = x_i + \tilde{h}_i$ is not, in the general case, a grid point, we deduce from Eq. (1.6) that

$$u_{t(\beta_1, \beta_2)} - \frac{\partial u}{\partial t}(\bar{x}, t) = O(\hbar^2 + \tau), \quad \bar{x} \in \hat{\omega}_h, \quad t \in \omega_\tau. \tag{2.1}$$

Using the conventional six-point stencil, we construct, on the basis of Eqs. (1.4) and (2.1), a monotonic scheme of second-order accuracy on the irregular grid $\omega = \hat{\omega}_h \times \omega_\tau$:

$$y_{t(\beta_1, \beta_2)} = 0.5[(k\hat{y})_{\bar{x}\hat{x}} + k_{(\beta_1, \beta_2)}\hat{y}_{\bar{x}\hat{x}} - k_{\bar{x}\hat{x}}\hat{y}_{(\beta_3, \beta_4)}] + \varphi, \tag{2.2}$$

$$y_i^0 = u_0(x_i), \quad u_0^{n+1} = \mu_1(t_{n+1}), \quad y_N^{n+1} = \mu_2(t_{n+1}). \tag{2.3}$$

Here, $v_t = (\hat{v} - v)/\tau$, $\hat{v} = v(x_i, t_{n+1})$, $\varphi = f(\bar{x}, \hat{t})$.

**Theorem 1.** *Suppose that the following conditions are satisfied:* $0 < k_1 \le k(x) \le k_2$,

$$\tau > \|h_+^2 - h^2\|_C/(6k_1). \tag{2.4}$$

*Then, the solution to the difference problem* (2.2), (2.3) *is stable with respect to the initial data, the boundary conditions, the right-hand side; and, for any* $t_n \in \omega_\tau$, *the a priory estimate*

$$\|y(t_{n+1})\|_{\bar{C}} \le \max\left\{\|y_0\|_{\bar{C}}, \max_{1 \le k \le n+1} \|y(t_k)\|_{C_\gamma}\right\} + \sum_{k=0}^{n} \tau\|\varphi(t_k)\|_C \tag{2.5}$$

*is valid.*

**Proof.** Let us represent scheme (2.2) in the canonical form (1.8):

$$A_i y_{i-1}^{n+1} - C_i y_i^{n+1} + B_i y_{i+1}^{n+1} = -F_i, \quad i = 1, 2, \ldots, N-1, \quad y_0^{n+1} = \mu_1(t_{n+1}), \quad y_N^{n+1} = \mu_2(t_{n+1}),$$

where

$$A_i = -\beta_{2i} + 0.5\tau[(k_{(\beta_1,\beta_2)} + k_{i-1})/(\hbar_i h_i) - \beta_{4i} k_{\bar{x}\hat{x},i}],$$

$$B_i = -\beta_{1i} + 0.5\tau[(k_{(\beta_1,\beta_2)} + k_{i+1})/(\hbar_i h_{i+1}) - \beta_{3i} k_{\bar{x}\hat{x},i}], \quad C_i = 1 + A_i + B_i, \quad F_i = y_{(\beta_1,\beta_2)} + \tau\varphi.$$

Let us verify conditions (1.10). In comparison with the stationary case considered above, the additional terms $-\beta_2$ and $-\beta_1$ appeared in expressions for the coefficients $A$ and $B$. Consider the case $\tilde{h} < 0$, for which $\beta_1 = 0$ and $-\beta_2 = \tilde{h}/h$. Neglecting the positive term with $k_{\bar{x}\hat{x}}$ in the expression for $A$, we obtain

$$A > \frac{\tilde{h}}{h} + 0.5\tau\frac{(1 + \tilde{h}/h)k - \tilde{h}k_-/h + k_-}{\hbar h} = \frac{\tilde{h}}{h} + 0.5\tau\frac{(1 + \tilde{h}/h)k + (1 - \tilde{h}/h)k_-}{\hbar h}.$$

Since $|\tilde{h}/h| < 1$, $A > (h_+ - h)/(3h) + 2\tau k_1/[h(h + h_+)]$. From the last inequality it follows that $A_i > 0$ for $\tau > |h_{i+1}^2 - h_i^2|/(6k_1)$. The coefficient $B$ is analyzed in a similar way. Thus, inequality (2.4) ensures the fulfillment of condition (1.10); therefore, following Lemma 1, we conclude that $\|y^{n+1}\|_{\bar{C}} \le \max\{\|y^{n+1}\|_{C_\gamma}, \|F\|_C\}$. Since the variable weighting factors $\beta_1$ and $\beta_2$ are nonnegative, $\|F\|_C \le \|y^n\|_{\bar{C}} + \tau\|\varphi^n\|_C$. Substituting this estimate into the last inequality, we obtain a series of the relations

$$\|y^{n+1}\|_{\bar{C}} \le \max\{\|y^{n+1}\|_{C_\gamma}, \|y^n\|_{\bar{C}} + \tau\|\varphi^n\|_C\}$$

$$\le \max\{\|y^{n+1}\|_{C_\gamma}, \|y^n\|_{C_\gamma} + \tau\|\varphi^n\|_C, \|y^{n-1}\|_C + \tau(\|\varphi^{n-1}\|_C + \|\varphi^n\|_C)\}$$

$$\le \ldots \le \max\left\{\max_{1 \le k \le n+1}\{\|y^k\|_{C_\gamma}\} + \sum_{k=0}^{n}\tau\|\varphi^k\|_C, \|y^0\|_{\bar{C}} + \sum_{k=0}^{n}\tau\|\varphi^k\|_C\right\}.$$

This inequality entails the required relationship. The theorem is proved.

**Remark 1.** If the grid is spatially regular ($h_+ = h$), then the difference scheme (2.2) is reduced to the strictly implicit scheme

$$y_t = (a\hat{y}_{\bar{x}})_x + \varphi, \quad a = 0.5(k + k_-), \tag{2.6}$$

for which the a priory stability estimate (2.5) is true without constraints (2.4) imposed on the relation between the time step $\tau$ and space step h of the grid (unconditional stability).

## 3. THE ENERGY INEQUALITY METHOD

To get rid of constraints (2.4), we invoke the general theory of operator difference schemes [1, 5]. Instead of (2.2), we will consider a more general scheme with the time-independent weighting factor $\sigma$

$$y_{(\beta_1,\beta_2)t} = (ay_{\bar{x}}^{(s)})_{\hat{x}} + 0.5[(h_+\beta_1 k_x - h\beta_2 k_{\bar{x}})y_{\bar{x}\hat{x}} + (h_+\beta_3 y_x - h\beta_4 y_{\bar{x}})k_{\bar{x}\hat{x}}] + \varphi, \quad a = 0.5(k + k_-). \tag{3.1}$$

Hereafter, we will suppose that $\mu_1 = \mu_2 = 0$ (homogeneous boundary conditions):

$$y_i^0 = u_0(x_i), \quad y_0^{n+1} = y_N^{n+1} = 0. \tag{3.2}$$

When formulating this difference scheme, we used the identity $v_{(\beta_1, \beta_2)} = v + h_+\beta_1 v_x - h\beta_2 v_{\bar{x}}$. For the sake of simplicity of the following calculations, we also suppose that $h_+ \geq h$ (the grid refines towards the origin of coordinates) and $k_{\bar{x}\hat{x}} \geq 0$ for all $x \in \hat{\omega}_h$. Then, the difference equation (3.1) can be rewritten in the form

$$Dy_t + A_1 y^{(\sigma)} + A_2 y = \varphi, \quad \varphi = f^{(\sigma)}(\bar{x}, t), \tag{3.3}$$

where $v^{(\sigma)} = \sigma v(t+\tau) + (1-\sigma)v(t) = \sigma\hat{v} + (1-\sigma)v$, and the operators $D, A_1, A_2$ for $i = 1, 2, \ldots, N-1$ are defined as follows:

$$\begin{gathered} D = E + A_0, \quad (A_0 y)_i = \tilde{h}_i y_{x,i}, \quad \tilde{h} = (h_+ - h)/3, \\ (A_1 y)_i = -(a y_{\bar{x}})_{\hat{x},i}, \quad (A_2 y)_i = -\tilde{h}_i(k_{x,i} y_{x,i} - 2k_{x,i} y_{\bar{x},i} + k_{\bar{x},i} y_{\bar{x},i})/(2\hbar_i). \end{gathered} \tag{3.4}$$

Let $\mathring{\Omega}_h$ be a set of grid functions defined for each $t \in \omega_\tau$ on the grid $\hat{\bar{\omega}}_h$ and vanishing on the boundary. Define the vector $y = y(t) = [y_1(t), \ldots, y_{N-1}(t)]^{\mathrm{T}}$ and define the linear space $H = \mathring{\Omega}_h$ to be a set of such vectors with the scalar product $(u, v) = \sum_{i=1}^{N-1} u_i v_i \hbar_i$ and the norm $\|\cdot\| = \sqrt{(\cdot, \cdot)}$. Since $A_1 = A_1^* > 0$, we denote by $H_{A_1}$ the Hilbert space which consists of the elements of the space $H$ and is equipped with the scalar product and the norm $\|y\|_{A_1}^2 = (A_1 y, y) = (a, y_{\bar{x}}^2] = \sum_{i=1}^{N-1} a_i y_{\bar{x},i}^2 h_i$. Then, the difference scheme (3.2), (3.3) can be written in the canonical form of two-layer operator difference schemes (see [1])

$$By_t = Ay = \varphi, \quad y_0 = u_0, \tag{3.5}$$

$$B = D + \sigma\tau A_1, \quad A = A_1 + A_2, \quad A_1 = A_1^* > 0. \tag{3.6}$$

We will use the following lemmas.

**Lemma 2** (see [4, 5, p. 348]). *Under arbitrary relations between the grid steps and for $h_+ \geq h$, the operator inequality $D \geq (1/3)E$ holds.*

**Lemma 3.** *Let in the operator difference scheme* (3.5), (3.6) *the uniform operator $A = A_1 + A_2$, $A_1 = A_1^* > 0$ and the operator $A_2$ be subordinated to the operator $A_1$ in the sense*

$$\|A_2 y\|^2 \leq \delta\|y\|_{A_1}^2, \quad \delta = \text{const} > 0. \tag{3.7}$$

*Then, for*

$$B(t) \geq \varepsilon E + 0.5\tau A_1, \quad \varepsilon > 0 \textit{ is an arbitrary constant}, \tag{3.8}$$

*where $\varepsilon$ is an arbitrary constant, the difference scheme is $\rho$-stable in $H_{A_1}$ and the a priory estimate*

$$\|y_{n+1}\|_{A_1}^2 \leq M\left(\|y_0\|_{A_1}^2 + 0.5\varepsilon^{-1}\sum_{k=0}^{n}\tau\|\varphi_k\|^2\right) \tag{3.9}$$

*holds, where the constant $M = e^{m_\varepsilon t_{n+1}}$ and $m_\varepsilon = \delta/(2\varepsilon)$.*

**Proof.** Taking into account the identity $y = y^{(0.5)} - 0.5\tau y_t$, we write Eq. (3.5) in the form $(B - 0.5\tau A_1)y_t + A_1 y^{(0.5)} = \varphi - A_2 y$. Multiply it scalarly in $H$ by $2\tau y_t$ to obtain

$$2\tau((B - 0.5\tau A_1)y_t, y_t) + \|y_{n+1}\|_{A_1}^2 = \|y_n\|_{A_1}^2 + 2\tau(\varphi - A_2 y, y_t). \tag{3.10}$$

Using the Cauchy inequality with respect to $\varepsilon$, we find the estimate $2\tau(\varphi - A_2 y, y_t) \leq \tau\varepsilon\|y_t\|^2 + 0.5\tau\varepsilon^{-1}(\|\varphi\|^2 + \|A_2 y\|^2)$. Taking into account this inequality and conditions (3.7) and (3.8), we deduce from

(3.10) the estimate

$$\|y_{n+1}\|^2_{A_1} \le (1 + \tau m_\varepsilon)\|y_n\|^2_{A_1} + 0.5\varepsilon^{-1}\tau\|\varphi_n\|^2 \le e^{\tau m_\varepsilon}(\|y_n\|^2_{A_1} + 0.5\varepsilon^{-1}\tau\|\varphi_n\|^2).$$

This entails the required inequality (3.9). The lemma is proved. Note, that a similar estimate is obtained in [10, p. 185] for nonself-adjoint operator $A$.

The following lemma establishes the subordination of operators $A_1$ and $A_2$ defined by formulas (3.4).

**Lemma 4.** *Suppose that in the difference scheme* (3.3) *the inequalities* $h_+/h \le c_1$ *and* $|k'| \le c_2$ *hold true. Then, inequality* (3.7) *is fulfilled with the constant* $\delta = 2c_2^2(1 + c_1)/k_1$.

**Proof.** According to the condition of the lemma,

$$|k_{\bar{x}}| \le h^{-1}\int_{x_-}^{x} |k'(\xi)|d\xi \le c_2.$$

Hence, the following estimate is valid for the norm $\|A_2 y\|$:

$$\|A_2 y\|^2 \le 2c_2^2(\|y_x\|^2 + \|y_{\bar{x}}^2\|). \tag{3.11}$$

Using the inclusion (see [5, p. 174]) $\|y_{\bar{x}}]|^2 \le k_1^{-1}\|y\|^2_{A_1}$ and the inequalities $\|y_x\|^2 \le \Sigma_{i=1}^N h_i y^2_{\bar{x},i} \le k_1^{-1}\|y\|^2_{A_1}$, $\|y_{\bar{x}}\|^2 \le c_1 k_1^{-1}\|y\|^2_{A_1}$, we deduce the required estimate from Eq. (3.11). The lemma is proved.

The auxiliary results presented above enable us to formulate the following main proposition of this section.

**Theorem 2.** *Let the conditions of Lemma* 2 *and Lemma* 4 *be fulfilled. Then, for* $\sigma \ge 0.5$ *and* $0 < \varepsilon \le 1/3$, *the difference scheme is stable with respect to the initial data and the right-hand side, and the a priory estimate* (3.9) *is valid for its solution.*

**Proof.** Let us verify condition (3.8). By virtue of the theorem assumptions, $B - (\varepsilon E + 0.5\tau A) = (1/3 - \varepsilon)E + (\sigma - 0.5)\tau A_1 \ge 0$. Therefore, estimate (3.9) holds. The theorem is proved.

**Remark 2.** We note the principal possibility of constructing difference schemes of third-order accuracy on an irregular grid using a three-point stencil. For simplicity, let $k(x) \equiv 1$. Then, for $\sigma_1 = -2(h + 2h_+)(h_+ - h)/(9\hbar h)$, $\sigma_2 = (2h + h_+)(h_+ - h)/(9\hbar h_+)$, and $\delta = (h^2 + hh_+ + h_+^2)/36$, the difference scheme

$$y_{(\sigma_1,\sigma_2)} + \delta y_{t\bar{x}\hat{x}} = y^{(0.5)}_{\bar{x}\hat{x}} + \varphi, \quad \varphi = f(\bar{x}, \bar{t}) + \delta f_{\bar{x}\hat{x}}, \quad \bar{t} = t_{n+1/2}, \tag{3.12}$$

approximates the initial problem to the order $O(\hbar^3 + \tau^2)$. Curiously, it was found that, in case of a regular grid, scheme (3.12) is reduced to the well-known scheme (see [1])

$$y_t = y^{(\sigma_0)}_{\bar{x}x} + \varphi, \quad \sigma_0 = 0.5 - h^2/(12\tau)$$

with the approximation order $O(h^4 + \tau^2)$.

## 4. NONLINEAR PROBLEM AND COMPUTATIONAL EXPERIMENT

Consider the quasilinear heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) + f(u), \quad -l < x < l, \quad 0 < t \le T,$$
$$u(x, 0) = u_0(x), \quad u(-l, t) = \mu_1(t), \quad u(l, t) = \mu_2(t). \tag{4.1}$$

The corresponding difference scheme of the second-order approximation with respect to the space variable has the form

$$y_{t(\beta_1,\beta_2)} = 0.5[(k(\hat{y})\hat{y})_{\bar{x}\hat{x}} + k_{(\beta_1,\beta_2)}(\hat{y})\hat{y}_{\bar{x}\hat{x}} - k_{\bar{x}\hat{x}}(\hat{y})\hat{y}_{(\beta_3,\beta_4)}] + \varphi(\hat{y}), \tag{4.2}$$

where $\varphi = f(\hat{\bar{y}})$, $\bar{y}_i = (y_{i-1} + y_i + y_{i+1})/3$. To demonstrate the effectiveness of the proposed algorithms, we compared calculations of the blowup $S$-mode combustion (see [11, p. 174]) by scheme (4.2) and by the non-

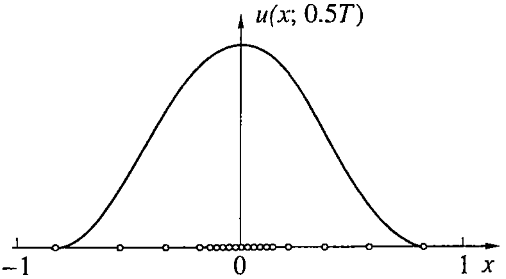


**Fig. 1.**

linear analog of the well-known conservative scheme (2.6) of the second-order accuracy and the first-order approximation

$$y_t = (a(\hat{y})\hat{y}_{\bar{x}})_{\hat{x}} + f(\hat{y}), \quad a_i = [k(y_i) + k(y_{i-1})]/2. \tag{4.3}$$

For solving nonlinear difference equations (4.2) and (4.3), we implemented an iterative procedure that can be written for Eq. (4.2) in the form

$$\frac{\overset{s+1}{\hat{y}}_{(\beta_1,\beta_2)} - y_{(\beta_1,\beta_2)}}{\tau} = 0.5[(k(\overset{s}{\hat{y}})\overset{s+1}{\hat{y}})_{\bar{x}\hat{x}} + k_{(\beta_1,\beta_2)}(\overset{s}{\hat{y}})\overset{s+1}{\hat{y}}_{\bar{x}\hat{x}} - k_{\bar{x}\hat{x}}(\overset{s}{\hat{y}})\overset{s+1}{\hat{y}}_{(\beta_3,\beta_4)}] + \varphi(\overset{s}{\hat{y}}),$$

where $s$ is the iteration number, $s = 0, 1, 2, \ldots$ . In the case of the $S$-mode combustion, $k(u) = u^\sigma$, $f(u) = u^{\sigma+1}$, $\sigma > 0$, and the self-similar solution in the domain $R \times (0, T_0)$ has the form

$$u(x, t) = \begin{cases} (T_0 - t)^{-1/\sigma}\left(\dfrac{2(\sigma+1)}{\sigma(\sigma+2)}\cos^2\dfrac{\pi x}{L_S}\right)^{1/\sigma}, & |x| < L_S/2, \\ 0, \quad |x| \ge L_S/2. \end{cases}$$

The following values of the parameters were used: $\sigma = 1$, $T_0 = 3$, $L_S = 8.886$, $-0.6L_S \le x \le 0.6L_S$, $0 \le t \le T_0/3$. The experiment was performed on grids described in Table 1, which shows the corresponding computation errors

$$\|z\|_C = \|y - u\|_C = \max_{(x,t)\in\omega} |y(x,t) - u(x,t)|.$$

Figure 1 shows the initial space irregular grid covering the plot of the exact solution to problem (4.1) for $t = T/2$. The number of grid nodes was multiplied by dividing each interval into two parts according to the rule $x_{2i} = (0.375 + r)x_{i+1} + (0.625 - r)x_i$, where $r \in [0, 0.25)$ was a normally distributed random variable.

The numerical experiment demonstrates the improved accuracy of the new scheme on coarse grids with respect to both space and time. It is particularly remarkable that on the finest grid the error of scheme (4.2) is half as large as the error of scheme (4.3). A more accurate calculation by scheme (4.3) requires a smaller time step $\tau$.

**Table 1**

| $N$ | $h_{max}$ | $\tau = 0.1$ | | $\tau = 0.01$ | | $\tau = 0.001$ | |
|---|---|---|---|---|---|---|---|
| | | scheme (4.2) | scheme (4.3) | scheme (4.2) | scheme (4.3) | scheme (4.2) | scheme (4.3) |
| 20 | 1.814 | 0.01 | 0.033 | 0.008 | 0.016 | 0.01 | 0.011 |
| 40 | 1.06 | 0.013 | 0.017 | 0.004 | 0.006 | 0.005 | 0.005 |
| 80 | 0.54 | 0.014 | 0.015 | 0.0014 | 0.0024 | 0.0014 | 0.0014 |
| 160 | 0.27 | 0.014 | 0.015 | 0.0011 | 0.0016 | 0.0005 | 0.0005 |
| 320 | 0.14 | 0.015 | 0.015 | 0.0013 | 0.0014 | 0.00017 | 0.00036 |

## 5. DIFFERENCE SCHEMES FOR THE WAVE EQUATION

Consider the boundary value problem for the vibration equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) + f(x, t), \quad 0 < x < l, \quad 0 < t \le T, \tag{5.1}$$

$$u(x, 0) = u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = \bar{u}_0(x), \quad u(0, t) = u(l, t) = 0. \tag{5.2}$$

Using the conventional notation $y = y^n$, $\hat{y} = y^{n+1}$, $\check{y} = y^{n-1}$, $y_t = (\hat{y} - y)/\tau$, $y_{\bar{t}} = (y - \check{y})/\tau$, $y_{\bar{t}t} = (y_t - y_{\bar{t}})/\tau$, and $v^{(\sigma_1, \sigma_2)} = \sigma_1 \hat{v} + (1 - \sigma_1 - \sigma_2)v + \sigma_2 \check{v}$, by analogy with Eqs. (3.1), (3.3), we construct the following scheme of the second-order space approximation on the grid $\hat{\bar{\omega}}_h \times \bar{\omega}_\tau$ for problem (5.1), (5.2):

$$y_{\bar{t}t} + [(h^2/6)y_{\bar{t}t\bar{x}}]_{\hat{x}} + A_1 y^{(\sigma_1, \sigma_2)} = \varphi - A_2 y, \tag{5.3}$$

$$y(x, 0) = u_0(x), \quad y_t(x, 0) = \tilde{u}_0(x), \quad \hat{y}_0 = \hat{y}_N = 0, \tag{5.4}$$

where $\varphi = f^{(\sigma_1, \sigma_2)}(\bar{x}, t)$, $t \in \omega_\tau$, $\tilde{u}_0(x)$ are chosen such that the approximation error of the second initial condition is $O(\tau^2)$, and the operators $A_1, A_2$ are determined by relations (3.4). For $k(x) = \text{const}$, scheme (5.3) coincides with the scheme suggested in [3]. For sufficiently smooth functions, $u_{\bar{t}t} + [(h_2/6)u_{\bar{t}t\bar{x}}]_{\hat{x}} = u_{\bar{t}t} + [(h_+ - h)/3]u_{\bar{t}tx} + (h^2/6)u_{\bar{t}t\bar{x}\hat{x}} = \partial^2 u(\bar{x}, t)/\partial t^2 + (h^2/6)u_{\bar{t}t\bar{x}\hat{x}} + O(\hbar^2 + \tau^2)$; thus, $u_{\bar{t}t} + [(h^2/6)u_{\bar{t}t\bar{x}}]_{\hat{x}, i} - \partial^2 u(\bar{x}_i, t_n)/\partial t^2 = O(\hbar_i^2 + \tau^2)$. Taking into account the identity $v^{(\sigma_1, \sigma_2)} = v + \tau(\sigma_1 - \sigma_2)v_{\mathring{t}} + 0.5\tau^2(\sigma_1 + \sigma_2)v_{\bar{t}t}$ and Eqs. (1.3), (1.6), (2.1), we conclude that the relation $(A_1 u^{(\sigma_1, \sigma_2)} + A_2 u) - (\partial/\partial x)(k(\bar{x})\partial u(\bar{x}, t)/\partial x) = O(\hbar^2 + \tau)$ holds. Therefore, the difference scheme (5.3), (5.4) approximates the original differential problem to the second order with respect to the space variable. To obtain the corresponding a priory estimates, we write the scheme in the canonical form of the three-layer operator difference schemes [1, 5, 10]

$$Dy_{\bar{t}t} + By_{\mathring{t}} + A_1 y = \varphi_1, \quad \varphi_1 - \varphi - A_2 y, \tag{5.5}$$

where $D = E + A_0 + 0.5\tau^2(\sigma_1 + \sigma_2)A_1$ and $A_0 y = [h^2/6)y_{\bar{x}}]_{\hat{x}}$, $B = \tau(\sigma_1 - \sigma_2)A_1$.

**Lemma 5.** *Suppose that, in the operator difference scheme* (5.5) $A_1 = A_1^* > 0$, $D = D^* > 0$, $B \ge 0$, $D \ge 0.25(1 + \varepsilon)\tau^2 A_1$, $D^{-1} \le E$, and $\varepsilon = \text{const} > 0$. *Then, the scheme is stable with respect to the initial data and the right-hand side, and for its solution the a priory estimate*

$$\|y_{n+1}\|_{A_1} \le M\left(\|y(0)\|_{A_1} + \|y_t(0)\|_D + \sum_{k=1}^{n} \tau\|\varphi_1(t_k)\|\right), \quad M = \sqrt{\varepsilon^{-1}(1 + \varepsilon)} \tag{5.6}$$

*holds for any* $\varepsilon > 0$.

**Lemma 6** (see [1]). *Suppose that* $g_n \ge 0$ $(n = 1, 2, \ldots)$ *and* $f_n \ge 0$ $(n = 0, 1, \ldots)$ *are nonnegative functions. If* $f_n$ *is a nondecreasing function* $(f_{n+1} \ge f_n)$, *then it follows from the inequality* $g_{n+1} \le c_0 \Sigma_{k=1}^{n} \tau g_k + f_n$ $(n = 1, 2, \ldots, g_1 \le f_0, c_0 = \text{const} > 0)$ *that* $g_{n+1} \le \exp(c_0 t_n) f_n$.

On the basis of Lemma 5 and Lemma 6 and the subordination condition for operators (3.7) with the constant $\delta = 2c_2^2(1 + c_1)/k_1$ (see Lemma 4), we prove the following theorem.

**Theorem 3.** *Suppose that in the difference scheme* (5.3), (5.4) $\sigma_1 \ge \sigma_2$, $\sigma_1 + \sigma_2 \ge 0.5(1 + \varepsilon) + h_{\max}^2/(3\tau^2 k_1)$, $h_+ \ge h$, $0 < k_1 \le k(x) \le k_2$, $|k'| \le c_2$, $h_+/h \le c_1$, and $h_{\max} = \max_{1 \le i \le N} h_i$. *Then, for the solution of the difference problem, the a priory estimate* (5.6) *with the constant* $M = \sqrt{n_\varepsilon}\exp(\sqrt{\delta n_\varepsilon} t_n)$ *(where* $n_\varepsilon = (1 + \varepsilon)/\varepsilon$*) holds.*

**Proof.** Let us verify the conditions of Lemma 5. Since $\|[y_{\bar{x}}]\| \le k_1^{-1}\|y\|_{A_1}$, then $A_0 \ge -(h_{\max}^2/(6k_1))A_1$. Therefore, $D - 0.25(1 + \varepsilon)\tau^2 A_1 \ge E + 0.5\tau^2[(\sigma_1 + \sigma_2) - h_{\max}^2/(3\tau^2 k_1) - 0.5(1 + \varepsilon)]A_1 \ge 0$, $B \ge 0$, $A_1 = A_1^* >$

0, $D = D^* > 0$, and $D^{-1} \le E$ if the theorem conditions hold. On the basis of the a priory estimate (5.6), we conclude that $\|y_{n+1}\|_{A_1} \le \sqrt{n_\varepsilon}(\|y(0)\|_{A_1} + \|y_t(0)\|_D + \Sigma_{k=1}^n \tau\|\varphi_1(t_k)\|$. Since $\|\varphi_{1k}\| \le \|\varphi_k - A_2 y_k\| \le \|\varphi_k\| + \sqrt{\delta}\,\|y_k\|_{A_1}$ is deduced from the triangle inequality and the subordination conditions of operators (3.7), the following estimate holds:

$$\|y_{n+1}\|_{A_1} \le \sqrt{n_\varepsilon}\left(\|y_0\|_{A_1} + \|y_{t,0}\|_D + \sum_{k=1}^{n}\tau\|\varphi_k\|\right) + \sqrt{\delta n_\varepsilon}\sum_{k=1}^{n}\tau\|y_k\|_{A_1}.$$

Strengthening the latter inequality with the help of Lemma 6, we obtain the assertion of the theorem.

A drawback of scheme (5.3) considered above is that it is not reduced to the classical difference scheme for a wave equation on a regular grid. Furthermore, the strictly implicit scheme ($\sigma_1 = 1, \sigma_2 = 0$), according to Theorem 3, is stable for $\tau \ge \sqrt{2/(3k_1)}\,h$, which is opposite to the Courant condition. However, when reducing scheme (5.3) to the canonical form, the operator $D$ remains self-adjoint, and, in the case of constant coefficients, the corresponding difference scheme is conservative [2]. Below, we present another class of nondivergent schemes approximating the wave equation (5.1) on an arbitrary irregular space grid of order $O(\hbar^2 + \tau)$:

$$y_{\bar{t}t} + ((h_+ - h)/3)y_{\bar{t}t\bar{x}} = (ay_{\bar{x}}^{(\sigma_1, \sigma_2)})_{\hat{x}} + 0.5[(h_+\beta_1 k_x - h\beta_2 k_{\bar{x}})y_{\bar{x}\hat{x}} + (h_+\beta_3 y_x - h\beta_4 y_{\bar{x}})k_{\bar{x}\hat{x}}] + \varphi. \quad (5.7)$$

Let us analyze the stability of this difference scheme. For the sake of clarity of the proof, suppose that $k(x) = 1$, $h_+ \ge h$, $\sigma_1 = 1$, $\sigma_2 = 0$, and $\varphi = 0$. Then, scheme (5.7) takes the simple form

$$y_{\bar{t}t} + [(h_+ - h)/3]y_{\bar{t}t\bar{x}} = \hat{y}_{\bar{x}\hat{x}}. \quad (5.8)$$

**Theorem 4.** *Under the assumptions made and for $\tau \ge \|h_+ - h\|_C/3$, the difference scheme is stable with respect to the initial data, and the estimate $\|y_{\bar{x}\hat{x}}^n\| \le \|u_{0\bar{x}\hat{x}}\| + \|\tilde{u}_{0\bar{x}}\|$ holds.*

**Proof.** Unfortunately, we cannot apply the general theory of operator difference schemes, because the operator $D$ (see canonical form (5.5)) is nonself-adjoint. Let us multiply Eq. (5.8) scalarly by $-2\tau y_{t\bar{x}\hat{x}}$. We obtain the energy identity

$$\left\|y_{\bar{t}x}^{n+1}\right\|^2 + \left\|y_{\bar{x}\hat{x}}^{n+1}\right\|^2 + \tau^2(\|y_{\bar{t}t\bar{x}}\|^2 + \|y_{t\bar{x}\hat{x}}\|^2) = \left\|y_{\bar{t}x}^{n}\right\|^2 + \left\|y_{\bar{x}\hat{x}}^{n}\right\|^2 + 2\tau(\tilde{h}y_{\bar{t}t\bar{x}}, y_{t\bar{x}\hat{x}}). \quad (5.9)$$

Taking into account the inequalities $h_+ \ge h$, $\tilde{h} \le \tau$, we apply to the last term in Eq. (5.9) the Cauchy inequality with $\varepsilon$ to obtain $2\tau(\tilde{h}y_{\bar{t}t\bar{x}}, y_{t\bar{x}\hat{x}}) \le \tau^2\|y_{\bar{t}\bar{x}\hat{x}}\|^2 + \|\tilde{h}\|_C^2\|y_{\bar{t}t\bar{x}}\|^2 \le \tau^2(\|y_{\bar{t}\bar{x}\hat{x}}\|^2 + \|y_{\bar{t}t\bar{x}}\|^2)$. Substituting this inequality into (5.9), we obtain the required result. The theorem is proved.

## 6. MULTIDIMENSIONAL PROBLEM

Suppose that it is required to find a continuous function $u(x, t)$, $x = (x_1, \dots, x_p)$ satisfying the initial boundary value problem

$$\frac{\partial u}{\partial t} = Lu + f(x, t), \quad (x, t) \in Q_T,$$
$$Lu = \sum_{\alpha=1}^{p} L_\alpha u, \quad L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x, t)\frac{\partial u}{\partial x_\alpha}\right), \quad 0 < k_1 \le k_\alpha \le k_2, \quad (6.1)$$

$$u(x, 0) = u_0(x), \quad x \in \bar{\Omega}, \quad u(x, t) = \mu(x, t), \quad x \in \Gamma \quad (6.2)$$

within the domain $\bar{Q}_T = \bar{\Omega} \cup [0, T]$, $\bar{\Omega} = \Omega \cup \Gamma$, $\Omega = \{0 < x_\alpha < l_\alpha, \alpha = \overline{1, p}\}$.

In the domain $\bar{\Omega}$, we introduce an arbitrary rectangular grid irregular along each direction:

$$\hat{\bar{\omega}}_h = \{x = (x_1^{i_1}, \dots, x_p^{i_p}), x_\alpha^{i_\alpha} = x_\alpha^{i_\alpha - 1} + h_\alpha^{i_\alpha}, i_\alpha = 1, 2, \dots, N_\alpha - 1, x_\alpha^0 = 0, x_\alpha^{N_\alpha} = l_\alpha, \alpha = 1, 2, \dots, p\}.$$

Let $\hat{\omega}_h$ denote the set of internal nodes of the grid $\hat{\bar{\omega}}_h$ and $\gamma_h$ denote the set of boundary nodes. The space-time grid in the domain $\bar{Q}_T$ is introduced in the conventional way:

$$\bar{\omega} = \hat{\bar{\omega}}_h \times \bar{\omega}_\tau, \quad \bar{\omega}_\tau = \{t_n = n\tau, n = 0, 1, \dots, N_0, \tau N_0 = T\} = \omega_\tau \cup \{T\}.$$

Further, we will use the index-free notation for the coordinates ($x_\alpha = x_\alpha^{i_\alpha}$, $x_{\alpha\pm} = x_\alpha^{i_\alpha \pm 1}$) and for the grid functions ($v = v_{i_1, \dots, i_p} = v(x_1^{i_1}, \dots, x_p^{i_p}, t_n)$, $v^{\pm 1\alpha} = v_{i_1, \dots, i_\alpha \pm 1, \dots, i_p}$, $\alpha = 1, 2, \dots, p$). Let us introduce the grid functions $y_\alpha = y_\alpha(x^\alpha, t)$ and $k_\alpha = k_\alpha(x^\alpha, t)$ that are calculated for the points

$$(x^\alpha, t) = (\bar{x}_1^{i_1}, \dots, \bar{x}_{\alpha-1}^{i_\alpha - 1}, x_\alpha^{i_\alpha}, \bar{x}_{\alpha+1}^{i_\alpha + 1}, \bar{x}_p^{i_p}, t_n), \quad \bar{x}_\alpha = x_\alpha + \tilde{h}_\alpha, \quad \tilde{h}_\alpha = (h_{\alpha+} - h_\alpha)/3.$$

Then, the difference scheme of the second-order local approximation based on the use of the vector additive schemes [8, 12] and spatially varying weighting factors [9] has the form

$$y_{(\beta_{1l}, \beta_{2l})t} = \sum_{\alpha=1}^{l} \hat{\Lambda}_\alpha \hat{y}_\alpha + \sum_{\alpha=l+1}^{p} \Lambda_\alpha y_\alpha + \varphi, \quad \varphi = f(\bar{x}, \hat{t}), \quad l = 1, 2, \dots, p, \tag{6.3}$$
$$\Lambda_\alpha u_\alpha = 0.5[(k_\alpha u_\alpha)_{\bar{x}_\alpha \hat{x}_\alpha} + k_{\alpha(\beta_{1\alpha}, \beta_{2\alpha})} u_{\alpha \bar{x}_\alpha \hat{x}_\alpha} - k_{\alpha \bar{x}_\alpha \hat{x}_\alpha} u_{\alpha(\beta_{3\alpha}, \beta_{4\alpha})}],$$

$$y_l(x^l, 0) = u_0(x^l) = u_{0l}, \quad \hat{y}_l|_{\gamma_h} = \mu(x^l, t) = \mu_l, \quad x^l \in \gamma_h, \quad l = 1, 2, \dots, p. \tag{6.4}$$

Here,

$$v_{\bar{x}_\alpha \hat{x}_\alpha} = (v_{x_\alpha} - v_{\bar{x}_\alpha})/\hbar_\alpha, \quad \hbar_\alpha = 0.5(h_{\alpha+} + h_\alpha), \quad v_{x_\alpha} = (v^{+1_\alpha} - v)/h_{\alpha+}, \quad v_{\bar{x}_\alpha} = (v - v^{-1_\alpha})/h_\alpha,$$

$$\bar{x} = (\bar{x}_1, \dots, \bar{x}_p), \quad v_{(\beta_{jl}, \beta_{(j+1)l})} = \beta_{jl} y_l^{+1_l} + (1 - \beta_{jl} - \beta_{(j+1)l}) y_l + \beta_{(j+1)l} y_l^{-1_l}, \quad j = 1, 3,$$

$$\beta_{1l} = 0.5(\tilde{h}_l + |\tilde{h}_l|)/h_{l+}, \quad \beta_{2l} = 0.5(|\tilde{h}_l| - \tilde{h}_l)/h_l,$$

$$\beta_{3l} = 0.5(\tilde{h}_l k_{l\bar{x}_l\hat{x}_l} - |\tilde{h}_l k_{l\bar{x}_l\hat{x}_l}|)/(k_{l\bar{x}_l\hat{x}_l} h_{l+}), \quad \beta_{4l} = -0.5(\tilde{h}_l k_{l\bar{x}_l\hat{x}_l} + |\tilde{h}_l k_{l\bar{x}_l\hat{x}_l}|)/(k_{l\bar{x}_l\hat{x}_l} h_l).$$

The system of difference equations (6.3), (6.4) is solved using the sweep method sequentially in each direction beginning with $l = 1$. As a result, we obtain $p$ values of the solution at the points $(x^\alpha, t)$, $\alpha = 1, 2, \dots, p$. Note that, in the case of a regular grid scheme, (6.3) is reduced to the multicomponent alternating directions scheme [13, 14], and, in the case of constant coefficients, this scheme coincides with that considered in [9]. Using the Taylor expansion

$$u(x^\alpha, t) = u(x, t) + \sum_{\substack{l=1 \\ l \neq \alpha}}^{p} \tilde{h}_l \frac{\partial u(x^\alpha, t)}{\partial x_l} + O(|h|^2), \quad |h|^2 = \sum_{\alpha=1}^{p} h_\alpha^2, \quad h_\alpha = \max_{1 \le i_\alpha \le N_\alpha} h_\alpha^{i_\alpha},$$

the approximate solution at the grid point $(x, t) \in \omega$ can be found with the help of the second-order approximation formula

$$y(x, t) = y_\alpha(x^\alpha, t) - 0.5 \sum_{\substack{l=1 \\ l \neq \alpha}}^{p} [(\tilde{h}_l + |\tilde{h}_l|) y_{\alpha x_l} + (\tilde{h}_l - |\tilde{h}_l|) y_{\alpha \bar{x}_l}]. \tag{6.5}$$

We now turn to the derivation of estimates of the solution to the difference problem (6.3), (6.4) in the norm $C$. Similarly to the one-dimensional case, we define the grid norms

$$\|\cdot\|_C = \max_{x \in \hat{\omega}_h} |\cdot|, \quad \|\cdot\|_{\bar{C}} = \max_{x \in \hat{\bar{\omega}}_h} |\cdot|, \quad \|\cdot\|_{C_\gamma} = \max_{x \in \gamma_h} |\cdot|.$$

**Theorem 5.** *Let the conditions*

$$k_\alpha = k_\alpha(x^\alpha) \text{ is independ of } t, \quad \alpha = 1, 2, \dots, p, \quad \tau > \|h_{\alpha+}^2 - h_\alpha^2\|_C/(6k_1) \tag{6.6}$$

*be satisfied.*

*Then, the difference scheme* (6.3), (6.4) *is stable with respect to the initial data, the right-hand side, and the boundary conditions; and, for the solution to this problem, the estimate*

$$\|y_l\|_{\bar{C}} \le \|u_{0l}\|_{\bar{C}} + t\left(\max\left\{\|\mu\|_{C_t}, \left\|\sum_{\alpha=1}^{p}\Lambda_\alpha u_{0\alpha} + \varphi(0)\right\|_C\right\} + \sum_{t'=\tau}^{t-\tau}\tau\|\varphi_{\bar{t}}(t')\|_C\right)$$

*holds for any* $l = 1, 2, \ldots, p$, $t \in \omega_\tau$, *where*

$$\|\mu\|_{C_t} = \max_{0\le t'\le t-\tau}\left\{\max_{1\le l\le p}\|\mu_{lt}(t')\|_{C_\gamma}\right\}, \quad l = 1, 2, \ldots, p.$$

**Proof.** Using the expression $\hat{y} = y + \tau y_t$, Eq. (6.3) for $l = 1$ can be written as $y_{(\beta_{11}, \beta_{21})t} = \tau\Lambda_1 y_{1t} + \Sigma_{\alpha=1}^{p}\Lambda_\alpha y_\alpha + \varphi$. Taking into account the fact that $\Sigma_{\alpha=1}^{p}\Lambda_\alpha y_\alpha = y_{(\beta_{1p}, \beta_{2p})\bar{t}} + \check{\varphi}$, the last expression is reduced to the form

$$y_{(\beta_{11}, \beta_{21})t} = \tau\Lambda_1 y_{1t} + y_{(\beta_{1p}, \beta_{2p})\bar{t}} + \tau\varphi_{\bar{t}}. \quad (6.7)$$

Similarly, for $l > 1$, one can obtain

$$y_{(\beta_{1l}, \beta_{2l})\bar{t}} = \tau\Lambda_l y_{l\bar{t}} + y_{(\beta_{1(l-1)}, \beta_{2(l-1)})\bar{t}}, \quad l = 2, 3, \ldots, p. \quad (6.8)$$

By inequality (6.6), the maximum principle (1.10) holds for the problems $y_{(\beta_{1l}, \beta_{2l})t} = \tau\Lambda_l y_{lt} + \Phi$, $y_{lt,0} = y_{lt,N_l} = \mu_{lt}$, $l = 1, 2, \ldots, p$ with respect to the grid function $y_{l\bar{t}}$. Therefore, on the basis of inequality $\|y_{(\beta_{1l}, \beta_{2l})t}\|_C \le \|y_{lt}\|_C$ and Lemma 1, the following estimates are deduced from Eqs. (6.7), (6.8):

$$\|y_{1t}(t)\|_{\bar{C}} \le \max\{\|\mu_{1t}(t)\|_{C_\gamma}, \|y_{p\bar{t}}(t)\|_{\bar{C}} + \tau\|\varphi_{\bar{t}}(t)\|_C\},$$

$$\|y_{p\bar{t}}\|_{\bar{C}} \le \max\{\|\mu_{p\bar{t}}\|_{C_\gamma}, \|y_{(p-1)\bar{t}}\|_{\bar{C}}\} \le \ldots \le \max\left\{\max_{2\le l\le p}\|\mu_{l\bar{t}}\|_{C_\gamma}, \|y_{1\bar{t}}\|_{\bar{C}}\right\}.$$

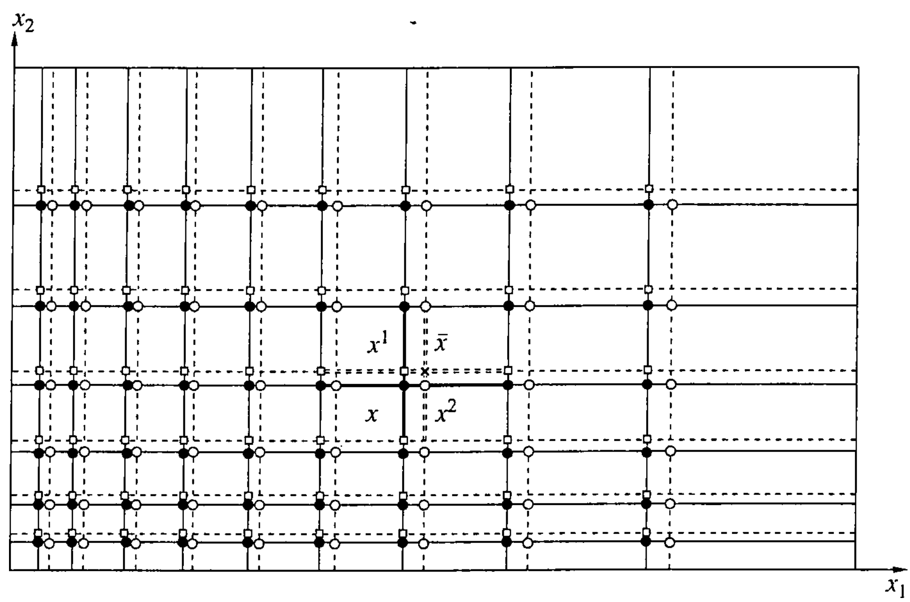


**Fig. 2.**

**Table 2**

| N | $h_{max}$ | $\tau = 0.1$ | | $\tau = 0.01$ | | $\tau = 0.001$ | |
|---|---|---|---|---|---|---|---|
| | | scheme (6.3) | scheme (6.9) | scheme (6.3) | scheme (6.9) | scheme (6.3) | scheme (6.9) |
| 10 | 0.384 | 1.55 | 2.318 | 0.491 | 0.69 | 0.431 | 0.405 |
| 20 | 0.214 | 0.547 | 1.557 | 0.142 | 0.376 | 0.13 | 0.13 |
| 40 | 0.13 | 0.15 | 0.626 | 0.04 | 0.135 | 0.036 | 0.045 |
| 80 | 0.056 | 0.064 | 0.588 | 0.013 | 0.115 | 0.01 | 0.02 |
| 160 | 0.034 | 0.021 | 0.4 | 0.004 | 0.077 | 0.0026 | 0.006 |

Hence, we find that

$$\|y_{1t}\|_{\bar{C}} \le \max\left\{\|\mu_{1t}\|_{C_\gamma},\ \max_{2\le l\le p}\|\mu_{l\bar{t}}\|_{C_\gamma},\ \|y_{1\bar{t}}\|_{\bar{C}}\right\} + \tau\|\varphi_{\bar{t}}(t)\|_C$$

$$\le \ldots \le \max\left\{\max_{\tau\le t'\le t}\|\mu_{1t}(t')\|_{C_\gamma},\ \max_{0\le t'\le t-\tau}\left\{\max_{2\le l\le p}\|\mu_{lt}(t')\|_{C_\gamma}\right\},\ \left\|\sum_{\alpha=1}^{p}\Lambda_\alpha u_{0\alpha} + \varphi(0)\right\|_C\right\} + \sum_{t'=\tau}^{t}\tau\|\varphi_{\bar{t}}(t')\|_C.$$

Substituting the last inequality into the obvious estimate $\|\hat{y}\|_{\bar{C}} \le \|y\|_{\bar{C}} + \tau\|y_t\|_{\bar{C}}$, we come to the assertion of the theorem.

**Remark 3.** In the case of a spatially regular grid, the difference scheme has the form [14]

$$y_t + \sum_{\alpha=1}^{l} A_\alpha \hat{y}_\alpha + \sum_{\alpha=l+1}^{p} A_\alpha y_\alpha = \varphi, \quad \varphi = f(x, \hat{t}), \quad y_\alpha = y_\alpha(x, t), \quad l = 1, 2, \ldots, p, \tag{6.9}$$

$$A_\alpha u_\alpha = -(a_\alpha u_{\alpha\bar{x}_\alpha})_{\hat{x}_\alpha}, \quad a_\alpha = 0.5(k_\alpha + k_\alpha^{-1_\alpha}), \quad k_\alpha = k_\alpha(x, t),$$

$$y_l(x, 0) = u_0(x), \quad \hat{y}_l|_{\gamma_h} = \mu(x, t), \quad x \in \gamma_h, \quad l = 1, 2, \ldots, p. \tag{6.10}$$

In this case, the assertion of Theorem 5 holds without constraints on the steps $\tau$ and $h$.

**Remark 4.** Applying, similarly to the one-dimensional case, the energy inequality method, one can prove the stability of scheme (6.3) in the space $W_2^1$ without constraints (6.6).

## 7. COMPUTATION OF THE TWO-DIMENSIONAL PROBLEM

To test the proposed scheme, we solved problem (6.1), (6.2) for $p = 2$, $k_l(x) = (x_l + \gamma_l)^{p_l}$, $l = 1, 2, \ldots, p$. The exact solution to this problem has the form

$$u(x, t) = \sum_{l=1}^{p}(x_l + \alpha_l)^{n_l} + \varepsilon(t + \beta)^m. \tag{7.1}$$

The right-hand side $f(x, t)$ is determined in accordance with the method of trial functions [1] by the direct substitution of the solution $u(x, t)$ and the coefficients $k_l(x)$ into the original equation. The calculations were carried out in a parallelepiped with the unit volume $T = 1$, $l_\alpha = 1$, $\alpha = 1, 2$. Since function (7.1) increases rapidly in the vicinity of the point $x = (x_1, \ldots, x_{l-1}, 0, x_{l+1}, \ldots, x_p)$ for $\alpha_l \ll 1$, $n_l < 0$, the initial irregular grid was refined in each direction towards the origin of the interval in the geometric progression (harmonic grids) $h_+ = qh$ with $q = 1.5$. The grid was refined as in the experiment described in Section 5. When solving the problem (6.1), (6.2), (7.1) we chose $\alpha_l = 0.1$, $n_l = -1$, $\gamma_l = 1$, $p_l = 1$, $l = 1, 2$, $m = 2$, $\beta = 1$, and $\varepsilon = 10^{-15}$. For comparison, the same problem was numerically solved using the first-order approximation scheme (6.9), (6.10). Table 2 presents the results of the calculations (see Section 5). Figure 2 illustrates the implementation of the efficient scheme (6.3) in the two-dimensional case. Here, we use the following notation: ●

denotes $x = (x_1, x_2)$, $\times$ denotes $\bar{x} = (\bar{x}_1, \bar{x}_2)$, $\square$ denotes $x' = (x_1, \bar{x}_2)$, and $\bigcirc$ denotes $x^2 = (\bar{x}_1, x_2)$. The scheme approximates the original equation to the second order with respect to the point $\bar{x}$. The grid functions $y_{1,2}$ are computed at points $(x^{1,2}, t)$. Then, we use Eq. (6.5) to determine the difference solution at the grid point $x$.

Note that the second order of space convergence of scheme (6.3) occurs already for $\tau = 0.1$, while schemes (6.9) have the same order only for $\tau = 0.001$. This yields a considerable reduction of the computation time required to obtain the prescribed accuracy.

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## REFERENCES

1. Samarskii, A.A., *Teoriya raznostnykh skhem* (The Theory of Difference Scheme), Moscow: Nauka, 1989.
2. Polevikov, V.K., Using Adaptive Grids for the Analysis of Free Surface in Statics of Ferrofluids, *Differ. Uravn.*, 1994, vol. 30, no. 12, pp. 2146–2152.
3. Samarskii, A.A., Vabishchevich, P.N., and Matus, P.P., Improved Accuracy Difference Schemes on Nonuniform Grids, *Differ. Uravn.*, 1996, vol. 32, no. 2, pp. 265–274.
4. Samarskii, A.A., Mazhukin, V.I., and Matus, P.P., Difference Schemes on Nonuniform Grids for the Two-dimensional Parabolic Equation, *Differ. Uravn.*, 1998, vol. 34, no. 77, pp. 980–987.
5. Samarskii, A.A., Vabishchevich, P.N., and Matus, P.P., *Raznostnye skhemy s operatornymi mnozhitelyami* (Difference Schemes with Operator Factors), Minsk: ZAO TsOTZh, 1998.
6. Samarskii, A.A., Vabishchevich, P.N., and Matus, P.P., Difference Schemes of Second-Order Accuracy on Nonuniform Grids, *Zh. Vychisl. Mat. Mat. Fiz.*, 1998, vol. 38, no. 3, pp. 413–424.
7. Samarskii, A.A., Mazhukin, V.I., Malafei, D.A., and Matus, P.P., Improving the Accuracy of Difference Schemes on Spatially Nonuniform Grids, *Dokl. Ross. Akad. Nauk,* 1999, vol. 367, no. 3, pp. 310–313.
8. Samarskii, A.A., Vabishchevich, P.N., and Matus, P.P., Stability of Vector Additive Schemes, *Dokl. Ross. Akad. Nauk,* 1998, vol. 361, no. 6, pp. 746–748.
9. Zyl', A.N. and Matus, P.P., Efficient Difference Schemes with Improved Accuracy for Multidimensional Parabolic Equations on Nonuniform Grids, *Zh. Vychisl. Mat. Mat. Fiz.*, 1999, vol. 39, no. 7, pp. 1151–1157.
10. Samarskii, A.A. and Gulin, A.V., *Ustoichivost' raznostnykh skhem* (Stability of Difference Schemes), Moscow: Nauka, 1973.
11. Samarskii, A.A., Galaktionov, V.A., Kurdyumov, S.P., and Mikhailov, A.P., *Rezhimy s obostreniem v zadachakh dlya kvazilineinykh parabolicheskikh uravnenii* (Blow-up Modes in Problems for Quasilinear Parabolic Equations), Moscow: Nauka, 1987.
12. Vabishchevich, P.N., Vector Additive Schemes for Evolutionary First-Order Equations, *Zh. Vychisl. Mat. Mat. Fiz.*, 1996, vol. 36, no. 3, pp. 44–51.
13. Abrashin, V.N. and Yakubenya, A.N., Efficient Schemes with Explicit Separation of the Front for Multidimensional Problems with Free Boundaries. I, *Differ. Uravn.*, 1990, vol. 26, no. 6, pp. 1055–1066.
14. Abrashin, V.N., A Variant of the Method of Alternating Directions for Multidimensional Problems in Mathematical Physics. I, *Differ. Uravn.*, 1990, vol. 26, no. 2, pp. 314–323.