Stability of Finite Difference S(emes on Non-uniform Spatial-Time-Grids

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Abstract. The three level operator finite difference schemes on non-uniform on time grids in Hilbert spaces of finite dimension are considered. A priori estimates for uniform stability on the initial conditions are received under natural assumptions on operators and non uniform time grids. The results obtained here are applied to study stability of the three levels weighted schemes of second order approximation $O\left(h^2+\tau_n\right)$ for some hyperbolic and parabolic equations of the second order. It is essential to note that the schemes of raised order of approximation are constructed here on standard stencils which are used in finite difference approximation techniques.

1 Introduction

Contemporaneous computational methods of mathematical physics alongside with the traditional requirements, such as stability and conservativity, have to satisfy also the adaptivity requirement. Application of adaptive grids first of all means that one have to use non uniform grid instead of uniform one which is adapted to behaviour of the singularities of the solution. It is known , that at use of non-uniform grids the order of local approximation becomes lower. One can increase the order of approximation by simple use of more extended stencils or by considering more restricted classes of solutions of differential problem. Let us to call attention to an another opportunity to increase accuracy expanding the approximation of initial differential equations from the points of a computational grid to some intermediate points of computational domain [1]. At present computational methods on non-uniform spatial grids have been widely studied for wide class of equations of mathematical physics with preservation of the second order local approximation with respect to the spatial variable [1] —[6]. Nevertheless the theoretical aspects of the three-level schemes on non-uniform time grids are less investigated [7,8].

This communication is devoted to investigation of the three level operator finite difference schemes on non-uniform on time grids with the operators acting in Hilbert spaces of finite dimension. The stability on initial conditions is proved and also a priori estimations in grid energy norms are obtained. Examples of the three level finite diff(ıce schemes of the second order of local approximations on time and spatial variables for parabolic and hyperbolic equation of the second order are presented. Especially we emphasize, that increase of the order of local approximation on non-uniform grids is achieved without increases of a standard stencil of the finite difference scheme.

2 Three Level Operator Finite Difference Schemes

Let us consider real Hilbert space H of finite dimension of real valued functions defined on non-uniform time grid

$$\hat{\bar{\omega}}_\tau = \{t_n = t_{n-1} + \tau_n,\ n \in 1,2,\dots,N_0;\ t_0 = 0,\ t_{N_0} = T\} = \hat{\omega}_\tau \bigcup \{0,T\} \ .$$

We designate as $D(t)$, $B(t)$, $A : H \to H$ linear operators in H. Let us consider a Cauchy problem for homogeneous finite difference operator equation

$$Dy_{\bar{t}\hat{t}} + By_t + Ay = 0, \quad y_0 = u_0, \quad y_1 = u_1 \ , \tag{1}$$

where $y = y_n = y(t_n) \in H$ is the unknown function, and $u_0,\ u_1 \in H$ are given functions. Here and in the following index-economic notations are used:

$$y_{\bar{t}\hat{t}} = (y_t - y_{\bar{t}})/\tau^*, \quad y_t = (y_{n+1} - y_n)/\tau_{n+1}, \quad y_{\bar{t}} = (y_n - y_{n-1})/\tau_n,$$

$$\hat{y} = y_{n+1}, \quad \check{y} = y_{n-1}, \quad \tau^* = 0{,}5\,(\tau_{n+1} + \tau_n), \quad y_{\overset{\circ}{t}} = \frac{y_{n+1} - y_{n-1}}{\tau_n + \tau_{n+1}} \ .$$

Let us designate as H_{R_k}, where $R_k^* = R_k \geq 0$ a space with inner product $(y,v)_{R_k}$, $y, v \in H$, and with semi-norms $\|y\|^2_{R_k} = (R_k y, y)$. Let us suppose that the operators entering in the scheme (1) satisfy the following conditions :

$$D(t) = D^*(t) \geq 0, \quad B(t) \geq 0, \quad t \in \hat{\omega}_\tau, \quad A = A^* > 0 \ , \tag{2}$$

$$D(t+\tau) \leq D(t), \quad \frac{\tau_{n+1}}{\tau_{n+2}} \leq \frac{\tau_n}{\tau_{n+1}}, \quad B(t_n) \geq 0{,}5\tau_{n+1}A \ , \tag{3}$$

where $A(t) = A$ is a constant operator. Concerning conditions (3) we shall make some observations.

Remark 1. Usually in the theory of stability of the three level finite difference schemes [1] with the variable operator $D(t)$ its Lipschitz-continuity on variable t is required. However, if one studies the stability , for example, of the weighted three level scheme [2]

$$y_{\bar{t}\hat{t}} + Ay^{(\sigma_1,\sigma_2)} = 0, \quad y_0 = u_0, \quad y_1 = u_1 \ , \tag{4}$$

$$y^{(\sigma_1,\sigma_2)} = \sigma_1\hat{y} + (1 - \sigma_1 - \sigma_2)\,y + \sigma_2\check{y} \ , \tag{5}$$

than this requirement implies undesirable requirement of quasi-uniformity of the time grid

$$|\tau_{n+1} - \tau_n| \leq c_0\tau_n^2, \quad n = 1,2,\dots,N_0 - 1 \ . \tag{6}$$

Remark 2. The second restriction from (3) $\tau_{n+1}/\tau_{n+2} \leq \tau_n/\tau_{n+1}$ is not rigid. Really, let the steps of a grid are chosen satisfying the geometrical progression law $\tau_{n+1} = q\tau_n$. Then the given inequality is valid for any $q = \text{const} > 0$.

Before we formulate results, we shall give definition of stability of the finite difference scheme (1) in case of the linear operators $D,\ B,\ A$.

Definition 1. *The operator finite difference scheme (1) is called unconditionally stable on initial conditions if there exist positive constants $M_1 > 0,\ M_2 > 0$, independent of τ_n and $u_0 \in H, u_1 \in H$, such that for all sufficiently small $\tau_n^* \leq \tau_0$, $n = 1, 2, \ldots, N_0$, the solution of the Cauchy problem (1) satisfies the estimate*

$$\|y_{\bar{t},n}\|^2_{R_{1n}} + \|y_n\|^2_{R_{2n}} \leq M_1\|y_{\bar{t},1}\|^2_{R_{11}} + M_2\|y_1\|^2_{R_{21}} \ . \tag{7}$$

If the inequality (7) is valid for every τ_n, then the scheme is called absolutely stable and when $M_1 = 1,\ M_2 = 1$ — uniformly stable.

Let us prove the following affirmation.

Theorem 1. *Let us suppose that the conditions (2), (3) are valid. Then the finite difference scheme (1) is uniformly stable with respect to initial conditions and the following estimate is valid*

$$\|y_{\bar{t},n+1}\|^2_{D_{n+1}} + \|y_{n+1}\|^2_{R_{n+1}} \leq \|y_{\bar{t},1}\|^2_{D_1} + \|y_1\|^2_{R_1} \ , \tag{8}$$

where $R_n = 0{,}5(1 + \tau_n/\tau_{n+1})A$.

Proof. Considering inner product of both parts of the equation (1) with $2\tau^* y_t$ and using the first condition from (4), one have

$$2\tau^*\left(Dy_{\bar{t}\hat{t}}, y_t\right) = 2\tau^*\left(Dy_{\bar{t}\hat{t}}, 0{,}5\left(y_t + y_{\bar{t}}\right) + 0{,}5\tau^* y_{\bar{t}\hat{t}}\right) =$$

$$= \|y_t\|^2_D - \|y_{\bar{t}}\|^2_D + \tau^{*2}\|y_{\bar{t}\hat{t}}\|^2_D \geq \|y_{\bar{t},n+1}\|^2_{D_{n+1}} - \|y_{\bar{t},n}\|^2_{D_n} \ ,$$

$$2\tau^*\left(By_t, y_t\right) = 2\tau_n^*\|y_{t,n}\|^2_{B_n} \ .$$

If the second condition from (3) is satisfied then one has inequality $\tau_n^*/\tau_{n+1} \geq \tau_{n+1}^*/\tau_{n+2}$. Therefore using the last estimate one obtain

$$2\tau^*\left(Ay, y_t\right) = \frac{\tau_n^*}{\tau_{n+1}}\left(\|y_{n+1}\|^2_A - \|y_n\|^2_A\right) - \tau_{n+1}\tau_n^*\|y_{t,n}\|^2_A \geq$$

$$\geq \|y_{n+1}\|^2_{R_{n+1}} - \|y_n\|^2_{R_n} - 2\tau_n^*\|y_{t,n}\|^2_{0,5\tau_{n+1}A} \ .$$

Summing these estimates and using the third condition from (3), one has the following relation

$$\|y_{\bar{t},n+1}\|^2_{D_{n+1}} + \|y_{n+1}\|^2_{R_{n+1}} \leq \|y_{\bar{t},n}\|^2_{D_n} + \|y_n\|^2_{R_n} \ ,$$

which is valid for every $n = 1, \ldots, N_0 - 1$. This immediately implicates the desired estimate (8). □

Example 1. Let us consider weighted three level operator finite difference scheme (4). Using the identities

$$y^{(\sigma_1,\sigma_2)} = y_n + (\sigma_1\tau_{n+1} - \sigma_2\tau_n)\, y_t + \sigma_2\tau_n\tau_n^* y_{\bar{t}t} \tag{9}$$

this scheme can be reduced to its canonical form (1) with

$$D_n = E + \tau_n\tau_n^*\sigma_2 A, \quad B_n = (\sigma_1\tau_{n+1} - \sigma_2\tau_n)\, A.$$

One can note that the conditions of the Theorem 1

$$D_{n+1} - D_n = \sigma_2\left(\tau_{n+1}\tau_{n+1}^* - \tau_n\tau_n^*\right) A \le 0,$$

$$B_n - 0{,}5\tau_{n+1}A = (\tau_{n+1}(\sigma_1 - 0{,}5) - \tau_n\sigma_2)\, A \ge 0$$

are satisfied if

$$\sigma_2\tau_{n+1}\tau_{n+1}^* \le \sigma_2\tau_n\tau_n^*, \quad \sigma_1 \ge \frac{1}{2} + \frac{\tau_n}{\tau_{n+1}}\sigma_2 \ .$$

On the harmonic grid $\tau_{n+1} = q\tau_n$ the first of above inequalities for $\sigma_2 > 0$ is satisfied on condensing grid with $q \le 1$, and for $\sigma_2 < 0$ this inequality is satisfied on dilating grid. If $\sigma_2 = 0$, $\sigma_1 = \sigma$, then the scheme (4) could be transformed to the following form (with constant operator $D_n = E$)

$$y_{\bar{t}t} + Ay^{(\sigma)} = 0, \quad y_0 = u_0, \quad y_1 = u_1 \ . \tag{10}$$

Here $y^{(\sigma)} = \sigma y_{n+1} + (1-\sigma)y_n$. As the Theorem 1 affirms, its solution satisfies the a priori estimate

$$\|y_{\bar{t},n}\|^2 + \|y_n\|^2_{R_n} \le \|y_{\bar{t},1}\|^2 + \|y_1\|^2_{R_1}, \quad n = 1,2,\ldots,N_0 \ , \tag{11}$$

(here still $R_n = 0{,}5(1 + \tau_n/\tau_{n+1})A$, and $A^* = A > 0$ — is constant operator) if the conditions

$$\sigma \ge \frac{1}{2} + \frac{\tau_n}{\tau_{n+1}}\sigma_2, \quad \frac{\tau_n}{\tau_{n+1}} \ge \frac{\tau_{n+1}}{\tau_{n+2}} \ . \tag{12}$$

are satisfied.

Example 2. The second order of local approximation scheme on non-uniform time grid. In rectangle $\overline{Q}_T = \overline{\Omega} \times [0,T]$, $\overline{\Omega} = \{x :\ 0 \le x \le l\}$, $0 \le t \le T$ let us consider the first initial boundary value problem for one dimensional parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right), \quad 0 < x < l, \quad 0 < t \le T \ , \tag{13}$$

$$u(0,t) = u(l,t) = 0, \quad t > 0, \quad u(x,0) = u_0(x), \quad 0 \le x \le l \ , \tag{14}$$

where $0 < c_1 \le k(x) \le c_2$, $c_1, c_2 = \text{const}$. On uniform in space and time variable grid

$$\overline{\omega} = \overline{\omega}_h \times \hat{\overline{\omega}}_\tau, \quad \overline{\omega}_h = \{x_i = ih,\ i = 0,1,\ldots,N,\ hN = l\} \ ,$$

let us approximate the differential problem (13), (14) by the following finite difference sheme

$$y_t + 0{,}5\tau_+ y_{\bar{t}\hat{t}} = (a\hat{y}_{\bar{x}})_x, \quad \tau_+ = \tau_{n+1} \ , \tag{15}$$

$$\hat{y}_0 = \hat{y}_N = 0, \quad y(x,0) = u_0(x), \quad x \in \overline{\omega}_h \ . \tag{16}$$

Here

$$a = a_i = 0{,}5\,(k_{i-1} + k_i)\,, \quad k_i = k\,(x_i)\,, \quad y = y_i^n = y\,(x_i, t_n) \ ,$$

$$(ay_{\bar{x}})_x = \left(a_{i+1}y^n_{\bar{x},i+1} - a_i y^n_{\bar{x},i}\right)/h, \quad y_{\bar{x},i} = \left(y_i^n - y_{i-1}^n\right)/h \ .$$

It is easy to verify that at the node (x_i, t_{n+1}) the three level scheme (13), (16) approximates differential problem with the second order, that is

$$\psi_i^{n+1} = -u_{t,i} - 0{,}5\tau_{n+1}u_{\bar{t}\hat{t},i} + \left(au_{\bar{x}}^{n+1}\right)_{x,i} = O\left(h^2 + \tau_{n+1}^2\right) \ .$$

The scheme (13) is one generalization of well known asymptotically stable scheme [3, p.309] at non-uniform time grid

$$\frac{3}{2}y_t - \frac{1}{2}y_{\bar{t}} = (ay_{\bar{x}})_x \ .$$

The scheme (13), (16) could be transformed to operator finite difference scheme (1), by putting $y = y_n = (y_1^n, y_2^n, \ldots, y_{N-1}^n)$, $(Ay)_i = -(ay_{\bar{x}})_{x,i}$, $i = 1, \ldots, N-1$, $y_0 = y_N = 0$, $D_n = 0{,}5\tau_{n+1}E$, $B_n = E + \tau_{n+1}A$. In this example the space $H = H_h$ consists in grid functions which are defined on the grid $\overline{\omega}_h$ and which are equal to zero on the boundary. Scalar product and norm are defined by expressions:

$$(y,v) = \sum_{i=1}^{N-1} hy_iv_i, \quad \|y\| = \sqrt{(y,y)} \ .$$

The properties of the operator A are well investigated [3]. In particular, $A^* = A > \delta E$, $\delta = 8c_1/l^2$. Let's check up the conditions of Theorem 1. It is obvious, that $D_n \le D_{n-1}$ pri $\tau_{n+1} \le \tau_n$, $B_n - 0{,}5\tau_{n+1} = E + 0{,}5\tau_{n+1}A > 0$ for every $\tau_{n+1} > 0$. Hence, the scheme (13), (16) is uniformly stable on initial data if

$$\frac{\tau_{n+1}}{\tau_{n+2}} \le \frac{\tau_n}{\tau_{n+1}}, \quad \tau_{n+1} \le \tau_n \ . \tag{17}$$

Let us note, that the conditions(17) are satisfied on harmonic grid $\tau_{n+1} = q\tau_n$ with arbitrary $0 < q \le 1$.

3 Finite Difference Schemes of Raised Order of Approximation on Non-uniform on Time and Space Grids

Suppose that in the domain $\overline{Q}_T$ it is required to find continuous function $u(x,t)$, satisfying following initial boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad 0 < t \le T \ ,$$

$$u(0,t) = u(l,t) = 0, \quad u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = \overline{u}_0(x) \ .$$

Let us consider next non-uniform spatial - time grid $\overline{\omega} = \hat{\overline{\omega}}_h \times \hat{\overline{\omega}}_\tau$:

$$\hat{\overline{\omega}}_h = \{x_i = x_{i-1} + h_i, \quad i = 1,2,\ldots,N, \quad x_0 = 0, \ x_N = l\} =$$
$$= \hat{\omega}_h \bigcup \{x_0 = 0, \ x_N = l\} \ ,$$
$$\hat{\overline{\omega}}_\tau = \{t_n = t_{n-1} + \tau_n, \quad n = 1,2,\ldots,N_0, \quad t_0 = 0, \ t_{N_0} = T\} =$$
$$= \hat{\omega}_\tau \bigcup \{t_0 = 0, \ t_{N_0} = T\} \ .$$

We approximate on this grid the differential problem by the finite difference one

$$y_{\bar{t}\hat{t}} + \frac{h_+ - h}{3} y_{\bar{t}\hat{t}\bar{x}} = y_{\bar{x}\hat{x}}^{(\sigma_1,\sigma_2)} \ , \tag{18}$$

$$y_0^{n+1} = y_N^{n+1} = 0, \quad y_i^0 = u_i^0, \quad y_{t,i}^0 = \tilde{u}_{0i}, \quad i = 0,1,\ldots,N \ . \tag{19}$$

Let us note, that $\tilde{u}_0(x)$, $x \in \hat{\overline{\omega}}_h$, is chosen in such a way that the error of approximation of the second initial condition has order $O(\tau_1^2)$:

$$\tilde{u}_0(x) = \overline{u}_0(x) + 0{,}5\tau_1 u_0''(x) \ .$$

Here usual designations are used [1]:

$$h_+ = h_{i+1}, \quad h = h_i, \quad y = y_i^n = y(x_i,t_n), \quad y_{\bar{x}} = \left(y_i^n - y_{i-1}^n\right)/h_i \ ,$$

$$y_{\bar{x}\hat{x}} = (y_x - y_{\bar{x}})/\hbar, \ y_x = (y_+ - y)/h_+, \ y_\pm = y(x_{i\pm1},t_n), \ \hbar = 0{,}5(h_+ + h) \ .$$

Let us show, that in supplementary node $(\overline{x}_i, \overline{t}_n)$:

$$\begin{aligned} \overline{x}_i &= \frac{1}{3}(x_{i-1} + x_i + x_{i+1}) = x_i + \frac{h_{i+1} - h_i}{3} \ , \\ \overline{t}_n &= \frac{1}{3}(t_{n-1} + t_n + t_{n+1}) = t_n + \frac{\tau_{n+1} - \tau_n}{3} \ , \end{aligned} \tag{20}$$

with

$$\sigma_1\tau_{n+1} - \sigma_2\tau_n = \frac{\tau_{n+1} - \tau_n}{3} \tag{21}$$

the finite difference scheme (18), (19) approximates the differential problem with the second order $O(\hbar_i^2 + \tau_n^{*2})$. For this purpose we shall rewrite the residual ψ as

$$\psi = u_{\bar{x}\hat{x}}^{(\sigma_1,\sigma_2)} - \left(u_{\bar{t}\hat{t}} + \frac{h_+ - h}{3} u_{\bar{t}\hat{t}\bar{x}}\right) = \psi_1 + \psi_2 \ ,$$

$$\psi_1 = u_{\bar{x}\hat{x}}^{(\sigma_1,\sigma_2)} - \frac{\partial^2 \overline{u}}{\partial x^2}, \quad \psi_2 = \frac{\partial^2 \overline{u}}{\partial t^2} - \left(u_{\bar{t}\hat{t}} + \frac{h_+ - h}{3} u_{\bar{t}\hat{t}\bar{x}}\right) \ .$$

Here $\overline{u} = u(\overline{x}, \overline{t})$, $\overline{x} = x + (h_+ - h)/3$, $\overline{t} = t + (\tau_+ - \tau)/3$. Let us note that an advantage of the scheme (18)(19), (21) is that fact, that for uniform grids ω_h, ω_τ

(t.e. $\tau_+ = \tau$, $h_+ = h$) this scheme reduces to the classical scheme of the order $O(h^2+\tau^2)$ on a uniform grid [3]. We proceed with the analysis of ψ_1, ψ_2. Using the identity (9) and the weight conditions (21), , we conclude, that for any grid function $v(x_i,t_n)$ the following relation is valid

$$v^{(\sigma_1,\sigma_2)} = v + \frac{\tau_+ - \tau}{3} v_t + \sigma_2 \tau \tau^* v_{\bar{t}\hat{t}} = v\left(x, \bar{t}\right) + O\left(\tau^{*2}\right) \ . \tag{22}$$

Hence,

$$\psi_1 = \psi_3 + O\left(\tau^{*2}\right), \quad \psi_3 = u_{\bar{x}\hat{x}}\left(x_i, \bar{t}\right) - \frac{\partial^2 \overline{u}}{\partial x^2} \ . \tag{23}$$

Using the Taylor series decomposition it is easy to show, that

$$\psi_3 = u_{\bar{x}\hat{x}}\left(x_i, \bar{t}_n\right) - \frac{\partial^2 u}{\partial x^2}\left(\overline{x}_i, \bar{t}_n\right) = \frac{\partial^2 u}{\partial x^2}\left(x_i, \bar{t}_n\right) + \frac{h_{i+1} - h_i}{3} \frac{\partial^3 u}{\partial x^3}\left(x_i, \bar{t}_n\right) - \\ - \frac{\partial^2 u}{\partial x^2}\left(\overline{x}_i, \bar{t}_n\right) + O\left(\hbar_i^2\right) = O\left(\hbar_i^2\right) \ . \tag{24}$$

By virtue of the next relations (which one can easy obtain with the help of the Taylor's formula)

$$v\left(x_i, t_n\right) + \frac{h_+ - h}{3} v_{\bar{x},i} = v\left(\overline{x}_i, t_n\right) + O\left(\hbar_i^2\right) \ ,$$

$$\frac{\partial^2 \overline{u}}{\partial t^2} - v_{\bar{t}\hat{t}}\left(\overline{x}_i, t_n\right) = O\left(\tau^{*2}\right) \ ,$$

one conclude, that the grid function ψ_2 is an infinitesimal of the second order , that is.

$$\psi_2 = O\left(\hbar_i^2 + \tau^{*2}\right) \ . \tag{25}$$

On the basis of the formulas (22) — (25) one conclude that finite difference scheme (18), (19), (21) approximates the initial boundary value problem for the wave equation on the standard 9-points stencil (see Fig. 1) with the second order (for sufficiently smooth function $u(x,t)$):

$$\psi = O\left(\hbar_i^2 + \tau_n^{*2}\right) \ .$$

For further investigation of the finite difference scheme (18), (19) some known formulas and identities are required:

$$y = \frac{\hat{y} + y}{4} + \frac{y + \check{y}}{4} - \frac{\tau_+ - \tau}{4} y_{\overset{\circ}{t}} - \frac{\tau\tau_+}{4} y_{\bar{t}\hat{t}} \ , \tag{26}$$

$$y^{(\sigma_1,\sigma_2)} = y + \left(\sigma_1 \tau_+ - \sigma_2 \tau\right) y_{\overset{\circ}{t}} + \frac{\sigma_1 + \sigma_2}{2} \tau\tau_+ y_{\bar{t}\hat{t}} \ , \tag{27}$$

$$y_{\overset{\circ}{t}} = \frac{y_t + y_{\bar{t}}}{2} + \frac{\tau_+ - \tau}{4} y_{\bar{t}\hat{t}} \ . \tag{28}$$

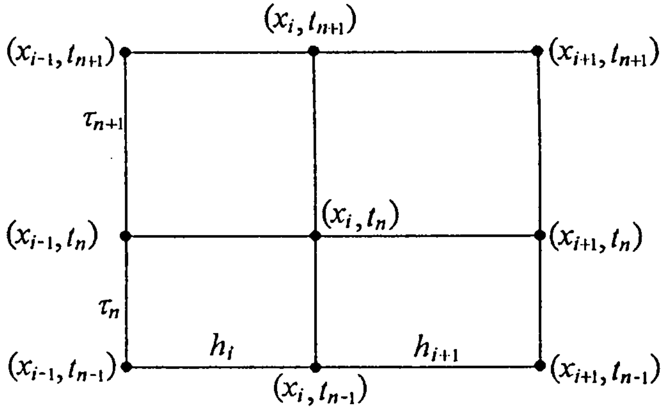

Fig. 1.

Let us introduce scalar products and norms of functions defined over a non-uniform spatial grid:

$$(y,v)_* = \sum_{i=1}^{N-1} \hbar_i y_i v_i, \quad \|y\|^2 = (y,y)_*, \quad (y,v] = \sum_{i=1}^{N} h_i y_i v_i, \quad \|y]| = (y,y] \ .$$

Lemma 1 **(First finite difference Green's formula).** *For any grid function $y(x)$, which is defined on non-uniform grid $\hat{\bar{\omega}}_h$ and vanish at $x=0$ and at $x=l$ the next formula is valid*

$$(y, v_{\bar{x}\hat{x}})_* = (y_{\bar{x}}, v_{\bar{x}}] \ . \tag{29}$$

One has the following theorem.

Theorem 2. *Let us suppose that*

$$\|\hbar/h\|_C \le c, \ \|\cdot\|_C = \max_{x\in\hat{\omega}_h} |\cdot|, \ \tau_{n+1}-\tau_n \ge \sqrt{\frac{2c}{3}}\, \|h_+ - h\|_C \,, \ n = 1,\dots,N_0-1 \ , \tag{30}$$

and

$$\sigma_1^n = \frac{2\tau_{n+1}+\tau_n}{6\,(\tau_{n+1}+\tau_n)}, \quad \sigma_2^n = \frac{\tau_{n+1}+2\tau_n}{6\,(\tau_{n+1}+\tau_n)} \ . \tag{31}$$

Then the finite difference scheme (18), (19) of the second order of local approximation $O(\hbar_i^2 + \tau^{*2})$ is uniformly stable and one has the estimation

$$||y_{t\bar{x}}^n]|^2 + \left\|y_{\bar{x}\hat{x}}^{(0,5)}(t_n)\right\|^2 \le ||y_{t\bar{x}}^0]|^2 + \left\|y_{\bar{x}\hat{x}}^{(0,5)}(0)\right\|^2 \ , \tag{32}$$

where $v^{(0,5)}(t_n) = 0,5(v^{n+1}+v^n), \ v_t^n = (v^{n+1}-v^n)/\tau_{n+1}$.

Proof. Let us note that σ_1^n, σ_2^n are defined with the formula (31) and satisfy the relation (21), which is necessary for increase the approximation order on a non-uniform grid. Let us multiply now the finite difference equation (18) by $-2\tau^*\hbar_i y_{\overset{\circ}{t}\bar{x}\hat{x},i}$ and sum at inner nodes of non-uniform space grid $\hat{\omega}_h$. After application the formula (29) we obtain the energy identity:

$$2\tau^*\left(y_{\bar{t}\hat{t}\bar{x}}, y_{\overset{\circ}{t}\bar{x}}\right] - 2\tau^*\left(\frac{h_+ - h}{3} y_{\bar{t}\hat{t}\bar{x}}, y_{\overset{\circ}{t}\bar{x}\hat{x}}\right)_* + 2\tau^*\left(y_{\bar{x}\hat{x}}^{(\sigma_1,\sigma_2)}, y_{\overset{\circ}{t}\bar{x}\hat{x}}\right)_* = 0 \ . \qquad (33)$$

Applying identity (28), one finds the equality

$$2\tau^*\left(y_{\bar{t}\hat{t}\bar{x}}, y_{\overset{\circ}{t}\bar{x}}\right] = ||y_{t\bar{x}}^n]|^2 - ||y_{t\bar{x}}^{n-1}]|^2 + 0{,}5\tau_n^*\left(\tau_{n+1} - \tau_n\right)||y_{\bar{t}\hat{t}\bar{x}}^n]|^2 \ , \qquad (34)$$

Using now formulas (26) (, 27) and condition of the second order approximations (21),we obtain for $y^{(\sigma_1,\sigma_2)}$ the following representation

$$y^{(\sigma_1,\sigma_2)} = \frac{1}{2}\left(y^{(0,5)} + \check{y}^{(0,5)}\right) + \frac{\tau_+ - \tau}{12} y_{\overset{\circ}{t}} \ . \qquad (35)$$

In deriving the formula (35) we used the property

$$\sigma_1^n + \sigma_2^n = \frac{1}{2} \ . \qquad (36)$$

Let us note, that if the variable weight multipliers do not satisfy to equality (36), then it is possible prove stability of the finite difference scheme (18),(19) only on quasi-uniform in time grid (6). In this case the estimation of stability will not carry uniform character, that is the constant $M_1 = \exp c_0 T$, appearing in definition 1 (see.(7)) will be much more than unit. Taking into account, that $y_{\overset{\circ}{t}} = (y^{(0,5)} - \check{y}^{(0,5)})/\tau^*$ and using (35) for third term in (34) one can find the following equality:

$$2\tau^*\left(y_{\bar{x}\hat{x}}^{(\sigma_1,\sigma_2)}, y_{\overset{\circ}{t}\bar{x}\hat{x}}\right)_* = \left\|y_{\bar{x}\hat{x}}^{(0,5)}(t_n)\right\|^2 - \left\|y_{\bar{x}\hat{x}}^{(0,5)}(t_{n-1})\right\|^2 + \tau_n^* \frac{\tau_{n+1} - \tau_n}{6}\left\|y_{\overset{\circ}{t}\bar{x}\hat{x}}\right\|^2 \qquad (37)$$

Using the algebraic inequality $2ab \geq -a^2 - b^2$, we shall estimate the last remaining scalar product in(33):

$$-2\tau^*\left(\frac{h_+ - h}{3} y_{\bar{t}\hat{t}\bar{x}}, y_{\overset{\circ}{t}\bar{x}\hat{x}}\right)_* \geq -\tau^* \frac{\tau_{n+1} - \tau_n}{2} ||y_{\bar{t}\hat{t}\bar{x}}]|^2 - $$
$$-\frac{2}{9}\left(\frac{(h_+ - h)^2}{\tau_{n+1} - \tau_n}\frac{\hbar_i}{h_i}, y_{\overset{\circ}{t}\bar{x}\hat{x}}^2\right)_* \ . \qquad (38)$$

Substituting obtained estimations (34) (,37) (,38) in energy identity (33), we come to recurrent relation

$$||y_{t\bar{x}}^n]|^2 + \left\|y_{\bar{x}\hat{x}}^{(0,5)}(t_n)\right\|^2 \leq ||y_{t\bar{x}}^{n-1}]|^2 + \left\|y_{\bar{x}\hat{x}}^{(0,5)}(t_{n_1})\right\|^2 \ ,$$

from which immediately follows the estimation (32). t_{n-1} □

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