PERGAMON Computers and Mathematics with Applications 44 (2002) 501–510

An International Journal
computers & mathematics with applications

www.elsevier.com/locate/camwa

Monotone Difference Schemes for Equations with Mixed Derivatives

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(Received and accepted May 2001)

Abstract—There are considered elliptic and parabolic equations of arbitrary dimension with alternating coefficients at mixed derivatives. For such equations, monotone difference schemes of the second order of local approximation are constructed. Schemes suggested satisfy the principle of maximum. *A priori* estimates of stability in the norm C without limitation on the grid steps τ and h_α, $\alpha = 1, 2, \dots, p$ are obtained (unconditional stability).

Keywords—Elliptic equations, Mixed derivatives, Maximum principle, Stability, Monotone difference scheme.

1. INTRODUCTION

For numerical solution of problems of mathematical physics, monotone finite difference schemes preserving second-order local approximation [1–3] are of particular interest because they are positively characterized from the practical point of view. Such algorithms have been developed for elliptic and parabolic equations (without mixed partial derivatives) which contain lower-order terms. When some monotone finite difference scheme is constructed, it is desirable to preserve the second order of approximation. For example, in [4,5], some nonlinear finite difference schemes of second order of local approximation in the spatial variable were developed for some nonlinear transfer equations. Note that maximum principle for finite element methods approximating the problems with mixed derivatives is considered in [6,7].

Here, combining two well-known difference schemes which possess the second order of approximation, one develops finite difference schemes satisfying maximum principle for multidimensional

0898-1221/02/$ - see front matter Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX
PII: S0898-1221(02)00164-5

elliptic and parabolic equations with mixed derivatives. For such algorithms, *a priori* estimates of stability in terms of the strong norm in C are obtained. We note that partial differential equations with mixed derivatives can appear when numerical algorithms for such classical equations as Laplace or Poisson equations are constructed on some nonorthogonal grid. Therefore, the schemes suggested here could be applied to development of effective numerical algorithms on nonorthogonal grids. In addition, they could be used as theoretical tools for investigations of stability and convergence of a dynamic adaptation method [8].

Development of obtained results for multidimensional problems of conjunction of hyperbolic and parabolic equations is of indubitable interest. Let us note that one-dimensional problems of conjunction without mixed derivatives have been considered in [9]. It should be emphasized that, for computational practice, investigations of computational algorithms for differential problems with generalized solutions are of more interest. In this case, in order to get the estimates of accuracy, it is more effective to use the method of energetic inequalities instead of the principle of maximum [10].

2. DIFFERENCE SCHEMES FOR EQUATIONS WITH COEFFICIENTS OF CONSTANT SIGNS

Let us consider in rectangle $\bar{G} = \{0 \le x_\alpha \le l_\alpha,\ \alpha = 1,2\}$ with the boundary Γ the problem of Dirichlet for an elliptic equation with mixed partial derivatives

$$Lu - q(x)u = -f(x), \qquad x \in G, \quad u = \mu(x), \quad x \in \Gamma, \quad x = (x_1, x_2), \quad q(x) \ge c_0 > 0, \tag{2.1}$$

$$Lu = \sum_{\alpha,\beta=1}^{2} L_{\alpha\beta} u, \qquad L_{\alpha\beta} u = \frac{\partial}{\partial x_\alpha}\left(k_{\alpha\beta}\frac{\partial u}{\partial x_\beta}\right). \tag{2.2}$$

One supposes that the following conditions of ellipticity are satisfied:

$$c_1 \sum_{\alpha=1}^{2} \xi_\alpha^2 \le \sum_{\alpha,\beta=1}^{2} k_{\alpha\beta}(x)\xi_\alpha\xi_\beta \le c_2 \sum_{\alpha=1}^{2} \xi_\alpha^2, \qquad x \in G, \tag{2.3}$$

where $c_1 > 0$, $c_2 > 0$ are some positive constants and $\xi = (\xi_1, \xi_2)$ is an arbitrary vector. From (2.3), we have that

$$0 < c_1 \le k_{\alpha\alpha} \le c_2, \qquad \alpha = 1,2, \qquad k_{11}k_{22} - k_{12}k_{21} \ge c_1^2.$$

In rectangle G, let us introduce a uniform grid

$$\bar{\omega}_h = \omega_h \cup \gamma_h,$$
$$\omega_h = \left\{x_i = \left(x_1^{(i_1)}, x_2^{(i_2)}\right),\ i_\alpha = 0,1,\dots,N_\alpha - 1,\ x_\alpha^{(0)} = 0,\ x_\alpha^{(N_\alpha)} = l_\alpha,\ \alpha = 1,2\right\},$$

with constant steps $h_1 = x_1^{(i_1)} - x_1^{(i_1-1)}$, $h_2 = x_2^{(i_2)} - x_2^{(i_2-1)}$; here γ_h denotes the set of all boundary points. Let us consider at the grid ω_h difference operators

$$\Lambda_{\alpha\alpha} y = \left(a_{\alpha\alpha} y_{\bar{x}_\alpha}\right)_{x_\alpha} = \frac{\left(a_{\alpha\alpha}^{(+1_\alpha)}\left(y^{(+1_\alpha)} - y\right) - a_{\alpha\alpha}\left(y - y^{(-1_\alpha)}\right)\right)}{h_\alpha^2},$$

$$\Lambda_{\alpha\beta}^{-} y = 0,5\left(\left(k_{\alpha\beta} y_{\bar{x}_\beta}\right)_{x_\alpha} + \left(k_{\alpha\beta} y_{x_\beta}\right)_{\bar{x}_\alpha}\right), \qquad \alpha,\beta = 1,2, \quad \alpha \ne \beta,$$
$$\Lambda_{\alpha\beta}^{+} y = 0,5\left(\left(k_{\alpha\beta} y_{x_\beta}\right)_{x_\alpha} + \left(k_{\alpha\beta} y_{\bar{x}_\beta}\right)_{\bar{x}_\alpha}\right), \qquad \alpha,\beta = 1,2, \quad \alpha \ne \beta,$$
$$a_{\alpha\alpha} = 0,5\left(k_{\alpha\alpha}\left(x_\alpha, x_{3-\alpha}\right) + k_{\alpha\alpha}\left(x_\alpha - h_\alpha, x_{3-\alpha}\right)\right), \qquad v^{(\pm 1_\alpha)} = v\left(x_\alpha \pm h_\alpha, x_{3-\alpha}\right).$$

In general, for difference approximation of equation (2.1), the following finite difference schemes of second order of local approximation are used [1]:

$$\Lambda^- y - dy = -\varphi, \quad x \in \omega_h, \qquad y = \mu(x), \quad x \in \gamma_h, \tag{2.4}$$

$$\Lambda^+ y - dy = -\varphi, \quad x \in \omega_h, \qquad y = \mu(x), \quad x \in \gamma_h, \tag{2.5}$$

where $\Lambda^- = \Lambda_{11} + \Lambda_{12}^- + \Lambda_{21}^- + \Lambda_{22}$, $\Lambda^+ = \Lambda_{11} + \Lambda_{12}^+ + \Lambda_{21}^+ + \Lambda_{22}$, $d \geq c_0$, φ are some stencil functionals of q and f, respectively [1].

To obtain *a priori* estimates of stability of the solution of the problem relative to the right side of equations and boundary conditions, we will use the maximum principle, therefore, we have to reduce difference schemes (2.4),(2.5) to canonical form (see [1, p. 242])

$$A(x)y(x) = \sum_{\xi \in \mathcal{M}'(x)} B(x,\xi)y(\xi) + F(x), \qquad x \in \omega_h, \quad y(x) = \mu(x), \quad x \in \gamma_h, \tag{2.6}$$

and to verify the following sufficient conditions:

$$A(x) > 0, \quad B(x,\xi) \geq 0, \quad D(x) = A(x) - \sum_{\xi \in \mathcal{M}'(x)} B(x,\xi) > 0, \qquad x \in \omega_h. \tag{2.7}$$

Here $\mathcal{M}'(x) = \mathcal{M}(x) \backslash \{x\}$, $\mathcal{M}(x)$ is the stencil of the scheme and the stencils of schemes (2.4),(2.5) are presented in Figures 1 and 2, respectively.

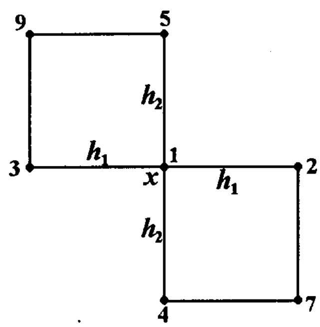

Figure 1. The stencil of scheme (2.4).

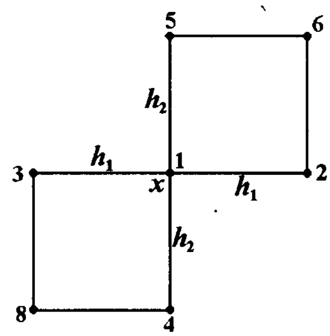

Figure 2. The stencil of scheme (2.5).

Let us number the stencil's nodes according to Figures 1 and 2. Then, for scheme (2.4), one obtains

$$\sum_{\xi \in \mathcal{M}'(x)} B(x,\xi) = \sum_{m=2}^{5} \overset{m}{B}{}^- + \overset{7}{B}{}^- + \overset{9}{B}{}^-,$$

and for scheme (2.5), one obtains

$$\sum_{\xi \in \mathcal{M}'(x)} B(x,\xi) = \sum_{m=2}^{6} \overset{m}{B}{}^+ + \overset{8}{B}{}^+,$$

respectively. The coefficients $\overset{m}{B}{}^+$ read

$$\overset{m}{B}{}^- = \frac{\overset{1}{k}_{11} + \overset{m}{k}_{11}}{2h_1^2} + \frac{\overset{1}{k}_{21} + \overset{m}{k}_{12}}{2h_1h_2}, \quad \overset{m}{B}{}^+ = \frac{\overset{1}{k}_{11} + \overset{m}{k}_{11}}{2h_1^2} - \frac{\overset{1}{k}_{21} + \overset{m}{k}_{12}}{2h_1h_2}, \qquad m = 2,3,$$

$$\overset{m}{B}{}^- = \frac{\overset{1}{k}_{22} + \overset{m}{k}_{22}}{2h_2^2} + \frac{\overset{1}{k}_{12} + \overset{m}{k}_{21}}{2h_1h_2}, \quad \overset{m}{B}{}^+ = \frac{\overset{1}{k}_{22} + \overset{m}{k}_{22}}{2h_2^2} - \frac{\overset{1}{k}_{12} + \overset{m}{k}_{21}}{2h_1h_2}, \qquad m = 4,5,$$

$$\overset{7}{B}{}^- = -\frac{\overset{2}{k}_{12}+\overset{4}{k}_{21}}{2h_1h_2}, \qquad \overset{6}{B}{}^+ = \frac{\overset{2}{k}_{12}+\overset{5}{k}_{21}}{2h_1h_2}, \qquad \overset{9}{B}{}^- = -\frac{\overset{3}{k}_{12}+\overset{5}{k}_{21}}{2h_1h_2}, \qquad \overset{8}{B}{}^+ = \frac{\overset{3}{k}_{12}+\overset{4}{k}_{21}}{2h_1h_2},$$

$$A^- = k + \frac{\overset{1}{k}_{11}}{h_1^2} + \frac{\overset{1}{k}_{12}+\overset{1}{k}_{21}}{h_1h_2} + \frac{\overset{1}{k}_{22}}{h_2^2} + d, \qquad k = \frac{\overset{2}{k}_{11}+\overset{3}{k}_{11}}{2h_1^2} + \frac{\overset{4}{k}_{22}+\overset{5}{k}_{22}}{2h_2^2} > 0,$$

$$A^+ = k + \frac{\overset{1}{k}_{11}}{h_1^2} - \frac{\overset{1}{k}_{12}+\overset{1}{k}_{21}}{h_1h_2} + \frac{\overset{1}{k}_{22}}{h_2^2} + d.$$

Using ellipticity condition (2.3) and supposing $\xi^- = (1/h_1, 1/h_2)$, $\xi^+ = (-1/h_1, 1/h_2)$, one can verify that the coefficients $A^\pm$ are positive. Analyzing the formulae for coefficients $\overset{m}{B}{}^\pm$, one can see that the difference scheme (2.4) should be used for negatives coefficients $k_{\alpha\beta}$, $\alpha \neq \beta$, while one can use scheme (2.5) for positives values of the coefficients.

For investigation of the stability of the solution regarding boundary conditions and right side simultaneously, the following result is more appropriate (particularly for nonstationary problems).

LEMMA 1. *Let us suppose that condition (2.7) of positivity of the coefficients of the difference scheme (2.6) is satisfied. Then the following estimation for the solution of problem (2.6) is valid:*

$$\|y\|_{\bar{C}} \le \max\left\{\|y\|_{C_\gamma}, \left\|\frac{F}{D}\right\|_C\right\}, \tag{2.8}$$

where

$$\|\cdot\|_{\bar{C}} = \max_{x\in\omega_h\cup\gamma_h} |\cdot|, \qquad \|\cdot\|_{C_\gamma} = \max_{x\in\gamma_h} |\cdot|, \qquad \|\cdot\|_C = \max_{x\in\omega_h} |\cdot|.$$

PROOF. The grid function may reach its maximum absolute value at the boundary of the region

$$\|y\|_{\bar{C}} \le \|y\|_{C_\gamma}, \tag{2.9}$$

or at one interior point. Let $|y(x)|$ reach its maximum value at some $x = x^* \in \omega_h$, so $|y(x^*)| \ge |y(x)|$ for each $x \in \bar{\omega}_h = \omega_h \cup \gamma_h$. Then, from equation (2.6) for $x = x^*$, there follows the inequality

$$A(x^*)\,|y(x^*)| \le \sum_{\xi\in\mathcal{M}'(x^*)} B(x^*,\xi)\,|y(x^*)| + |F(x^*)|.$$

Now one can find that

$$\left(A(x^*) - \sum_{\xi\in\mathcal{M}'(x^*)} B(x^*,\xi)\right)|y(x^*)| = D(x^*)\,|y(x^*)| \le |F(x^*)|.$$

Therefore,

$$\|y\|_{\bar{C}} \le \|y\|_C \le \left\|\frac{F}{D}\right\|_C.$$

The last estimation together with (2.9) proves the lemma.

The following affirmation is valid.

THEOREM 1. *Let us suppose that, for all $x \in \omega_h$, the following conditions of positivity of coefficients $\overset{m}{B}{}^\pm$, $m = 2,3,4,5$ are satisfied:*

$$\max_{m=4,5} \frac{\left|\overset{1}{k}_{12}+\overset{m}{k}_{21}\right|}{\overset{1}{k}_{22}+\overset{m}{k}_{22}} \le \frac{h_1}{h_2} \le \min_{m=2,3} \frac{\overset{1}{k}_{11}+\overset{m}{k}_{11}}{\left|\overset{1}{k}_{21}+\overset{m}{k}_{12}\right|}. \tag{2.10}$$

Then, for $k_{\alpha\beta}(x) \leq 0$, $\alpha \neq \beta$, the difference scheme (2.4) (and for $k_{\alpha\beta}(x) \geq 0$ the difference scheme (2.5) as well) is stable relative to the right side of equation and boundary conditions, and for its solution, the following estimation is valid:

$$\|y\|_{\bar{C}} \leq \max\left\{\frac{\|\mu\|_{C_\gamma}, \|\varphi\|_C}{c_0}\right\}. \tag{2.11}$$

PROOF. We will demonstrate the theorem for scheme (2.4). (For scheme (2.5), the demonstration is analogous.) As far as the coefficients $k_{12}(x)$, $k_{21}(x)$ are nonpositives for all $x \in \bar{G}$, then $\overset{7}{B}{}^- \geq 0$, $\overset{9}{B}{}^- \geq 0$. Using conditions (2.10), it is not difficult to demonstrate that all coefficients $\overset{m}{B}{}^- \geq 0$, $m = 2, 3, 4, 5$ are nonnegatives also. One could verify directly that for any node $x \in \omega_h$,

$$D(x) = A(x) - \sum_{\xi \in \mathcal{M}'(x)} B(x, \xi) = d(x) \geq c_0 > 0.$$

Now all conditions of Lemma 1 are satisfied and one can obtain the required inequality (2.8) from the *a priori* estimation (2.11). The theorem is demonstrated.

NOTE 1. If the matrix of the coefficients of equation (2.1) is diagonal dominated with respect to the rows and the columns $k_{\alpha\alpha} \geq |k_{\alpha\beta}|$, $\alpha, \beta = 1, 2$, $\alpha \neq \beta$, then one can choose $h_1 = h_2 = h$ which guarantees that condition (2.10) is always satisfied.

Let us consider now the problem of the convergence of the schemes under consideration. Let us suppose in what follows that the solution $u(x)$ and the coefficients (2.1) possess all bounded partial derivatives of respective orders. Substitution of $y = z + u$ in equations (2.4),(2.5) gives the next problems for the error function

$$\Lambda^- z - dz = -\psi^-, \qquad x \in \omega_h, \quad z = 0, \quad x \in \gamma_h, \tag{2.12}$$

$$\Lambda^+ z - dz = -\psi^+, \qquad x \in \omega_h, \quad z = 0, \quad x \in \gamma_h, \tag{2.13}$$

where $\psi^\mp = \Lambda^\mp u - du + \varphi$ denotes the error of approximation of difference schemes (2.4),(2.5) corresponding to exact solution of the differential problem (2.1). We will suppose that the stencil functionals $d(x)$ and $\varphi(x)$ satisfy the usual conditions of approximation of the coefficient $q(x)$ and right side $f(x)$:

$$d(x) - q(x) = O\left(h_1^2 + h_2^2\right), \qquad \varphi(x) - f(x) = O\left(h_1^2 + h_2^2\right). \tag{2.14}$$

It should be noted also (see [1, p. 268]), that for smooth solutions

$$\Lambda^- u - Lu = O\left(h_1^2 + h_2^2\right), \qquad \Lambda^+ u - Lu = O\left(h_1^2 + h_2^2\right).$$

Therefore, the difference schemes (2.4),(2.5) approximate the initial differential problem (2.1) up to second order, so we have

$$\left\|\psi^-\right\|_C \leq M\left(h_1^2 + h_2^2\right), \qquad \left\|\psi^+\right\|_C \leq M\left(h_1^2 + h_2^2\right),$$

where $M > 0$ is some positive constant which does not depend on the grid steps values h_1 and h_2.

Using Theorem 1 for the solutions of problems (2.12),(2.13), it can be verified that the following theorem is valid.

THEOREM 2. *Let us suppose that, for all $x \in \omega_h$, conditions (2.10) are satisfied. Then, for $k_{\alpha\beta} \leq 0$, $\alpha \neq \beta$, the solution of the difference scheme (2.4) (and for $k_{\alpha\beta} \geq 0$, $\alpha \neq \beta$, the solution of difference scheme (2.4) also) converges to the exact solution of the differential problem (2.1) and is valid for the next estimation:*

$$\|y - u\|_C \leq M c_0^{-1}\left(h_1^2 + h_2^2\right).$$

3. SCHEMES FOR EQUATIONS WITH COEFFICIENTS OF VARIABLE SIGNS

In this section, we will construct some monotone finite difference schemes for the numerical solution of the boundary value problem (2.1) with a nondivergent operator L in the form

$$Lu = k_{11}(x)\frac{\partial^2 u}{\partial x_1^2} + 2k_{12}(x)\frac{\partial^2 u}{\partial x_1 \partial x_2} + k_{22}(x)\frac{\partial^2 u}{\partial x_2^2}, \tag{3.15}$$

where the coefficients $k_{12}(x)$ may change sign for $x \in G$. In addition, one will suppose that the ellipticity condition (2.3) continues to be valid. One will approximate the differential problem (2.1),(3.15) with some monotone difference scheme of the second order of approximation $O(h_1^2 + h_2^2)$ which can be presented as

$$\Lambda y - dy = -\varphi, \qquad x \in \omega_h, \quad y = \mu(x), \quad x \in \gamma_h, \tag{3.16}$$

where

$$\Lambda y = k_{11} y_{\bar{x}_1 x_1} + k_{12}^{+} \Lambda_{12}^{+} y + k_{12}^{-} \Lambda_{12}^{-} y + k_{22} y_{\bar{x}_2 x_2}, \tag{3.17}$$

$$k_{12}^{+}(x) = 0,5\,(k_{12}(x) + |k_{12}(x)|) \geq 0, \quad \Lambda_{12}^{+} y = y_{\bar{x}_1 \bar{x}_2} + y_{x_1 x_2}, \qquad x \in \omega_h, \tag{3.18}$$

$$k_{12}^{-}(x) = 0,5\,(k_{12}(x) - |k_{12}(x)|) \leq 0, \quad \Lambda_{12}^{-} y = y_{\bar{x}_1 x_2} + y_{x_1 \bar{x}_2}, \qquad x \in \omega_h. \tag{3.19}$$

The stencil of scheme (3.16)–(3.19) generally involves nine nodes.

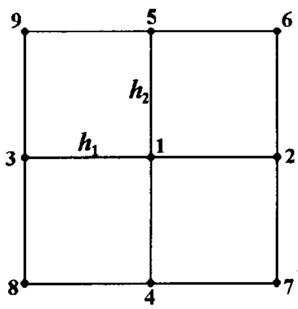

Figure 3. The stencil of the difference scheme (3.16)–(3.19).

The difference scheme (3.16)–(3.19) can be rewritten in canonical form (2.6) with the coefficients

$$\begin{gathered} \overset{2}{B} = \overset{3}{B} = \frac{k_{11}}{h_1^2} - \frac{|k_{12}|}{h_1 h_2}, \qquad \overset{4}{B} = \overset{5}{B} = \frac{k_{22}}{h_2^2} - \frac{|k_{12}|}{h_1 h_2}, \\ \overset{6}{B} = \overset{8}{B} = \frac{2k_{12}^{+}}{h_1 h_2} \geq 0, \qquad \overset{7}{B} = \overset{9}{B} = -\frac{2k_{12}^{-}}{h_1 h_2} \geq 0, \qquad A(x) = \sum_{m=1}^{9} \overset{m}{B} + d. \end{gathered} \tag{3.20}$$

It is easy to see that the coefficients (3.20) are positive for any $x \in G$ and

$$\frac{|k_{12}|}{k_{22}} \leq \frac{h_1}{h_2} \leq \frac{k_{11}}{|k_{12}|}. \tag{3.21}$$

For the system of inequalities (3.21) to make sense, it is necessary that the coefficients of equations (2.1),(3.15) satisfies the condition $k_{11}/|k_{12}| - |k_{12}|/k_{22} \geq 0$. But since from the ellipticity condition (2.3), one has the estimation

$$k_{11}k_{22} - k_{12}^2 \geq c_1^2 > 0,$$

then one could choose the steps h_1 and h_2 from relations (3.21). Now the difference scheme (3.16)–(3.19) is monotone and for its solution, the estimation in the form (2.11) for stability relative to the boundary conditions and right side is still valid.

4. DIFFERENCE SCHEMES FOR EQUATIONS OF GENERAL TYPE

Unfortunately, one cannot construct directly the monotone schemes for the divergent (conservative) form of equations (2.1),(2.2) with alternating signs coefficients $k_{12}(x)$, $k_{21}(x)$ of the mixed partial derivatives. Therefore, to obtain the above-mentioned results in their complete form, in this case, one has to consider the equations which contain lower-order derivatives. Let us consider again differential problem (2.1) with

$$Lu = \sum_{\alpha=1}^{2} \left(k_{\alpha\alpha}(x) \frac{\partial^2 u}{\partial x_\alpha^2} + k_\alpha(x) \frac{\partial u}{\partial x_\alpha} \right) + 2k_{12}(x) \frac{\partial^2 u}{\partial x_1 \partial x_2} - q(x)u, \tag{4.22}$$

where $|k_\alpha| \le c_3$, $q(x) \ge c_0 > 0$, and the coefficients $k_{\alpha\beta}(x)$, $\alpha, \beta = 1, 2$, are satisfying the ellipticity conditions (2.3). It is worth noting that the divergent operator (2.2) can always be transformed to the form (4.22). For the construction of the corresponding monotone scheme of the second order of local approximation $O(h_1^2 + h_2^2)$ for the equations with lower-order derivatives, we will use the ideas of Samarskii [1, p. 184]. We will approximate derivatives in (4.22) on a nonuniform grid ω_h using the finite difference relations

$$k_{\alpha\alpha}(x) \frac{\partial^2 u}{\partial x_\alpha^2} + k_\alpha(x) \frac{\partial u}{\partial x_\alpha} = \frac{k_{\alpha\alpha}}{1+R_\alpha} u_{\bar{x}_\alpha x_\alpha} + k_\alpha^+ u_{x_\alpha} + k_\alpha^- u_{\bar{x}_\alpha} + O\left(h_\alpha^2\right),$$

$$R_\alpha = 0,5|k_\alpha| \frac{h_\alpha}{k_{\alpha\alpha}}, \qquad k_\alpha^\pm = 0,5(k_\alpha \pm |k_\alpha|),$$

$$2k_{12}(x) \frac{\partial^2 u}{\partial x_1 \partial x_2} = k_{12}^+ \Lambda_{12}^+ u + k_{12}^- \Lambda_{12}^- u + O\left(h_1^2 + h_2^2\right).$$

It will result in the following difference scheme of the second order of local approximation:

$$\begin{gathered}\sum_{\alpha=1}^{2} \left(\frac{k_{\alpha\alpha}}{1+R_\alpha} y_{\bar{x}_\alpha x_\alpha} + k_\alpha^+ y_{x_\alpha} + k_\alpha^- y_{\bar{x}_\alpha} \right) + k_{12}^+ \Lambda_{12}^+ y + k_{12}^- \Lambda_{12}^- y - dy = \varphi, \qquad x \in \omega_h, \\ y = \mu(x), \qquad x \in \gamma_h,\end{gathered} \tag{4.23}$$

where the operators $\Lambda_{12\pm}$ are defined from the relations (3.18),(3.19), and d, φ are some stencil functionals [1], which can, in particular, be considered in the next form: $d(x) = q(x)$, $\varphi(x) = f(x)$, $x \in \omega_h$. In order for scheme (4.23) to be monotone (and therefore, to satisfy the maximum principle), it will be enough to require that the following conditions be satisfied:

$$\frac{|k_{12}(x)|\,(1+R_2(x))}{k_{22}(x)} \le \frac{h_1}{h_2} \le \frac{k_{11}(x)}{|k_{12}(x)|\,(1+R_1(x))}, \qquad x \in \omega_h. \tag{4.24}$$

As long as $R_\alpha(x) = O(h_\alpha)$, one can always satisfy the system of inequalities (4.24).

NOTE 2. One can naturally generalize all the above-obtained results to p-dimensional (here $p \ge 2$ is any real number) elliptic equations with mixed derivatives.

5. DIFFERENCE SCHEMES FOR MULTIDIMENSIONAL PARABOLIC EQUATIONS

Let us suppose now that $G = \{0 < x_\alpha < l_\alpha,\ \alpha = \overline{1,p}\}$ is a p-dimensional parallelepiped with the boundary Γ, $x = (x_1, x_2, \dots, x_p)$. One has to find a continuous function $u(x,t)$ which gives in $\bar{Q}_T = \bar{G} \times [0,T]$, a solution of the initial boundary value problem

$$\frac{\partial u}{\partial t} = Lu + f(x,t), \qquad x \in G, \qquad t \in (0,T], \qquad u(x,0) = u_0(x), \qquad u|_\Gamma = \mu(x,t), \tag{5.25}$$

$$Lu = \sum_{\alpha,\beta=1}^{p} L_{\alpha\beta} u, \qquad L_{\alpha\beta} u = \frac{\partial}{\partial x_\alpha} \left(k_{\alpha\beta}(x) \frac{\partial u}{\partial x_\beta} \right). \tag{5.26}$$

Let us suppose that the following conditions which are similar (2.3) are satisfied:

$$c_1 \sum_{\alpha=1}^{p} \xi_\alpha^2 \le \sum_{\alpha,\beta=1}^{p} k_{\alpha\beta}(x)\xi_\alpha\xi_\beta \le c_2 \sum_{\alpha=1}^{p} \xi_\alpha^2, \qquad x \in G. \tag{5.27}$$

Here, to simplify calculations without loss of generality, one can suppose that the coefficients depend only on spatial variables and that the coefficients of mixed derivatives have the same sign. Now, from (5.27), one has

$$0 < c_1 \le k_{\alpha\alpha}(x) \le c_2, \quad \alpha = 1, 2, \qquad k_{\alpha\beta}(x) \ge 0, \quad \alpha, \beta = \overline{1,p}, \quad \alpha \ne \beta, \quad x \in G. \tag{5.28}$$

Let us introduce in interval $[0, T]$ a uniform grid $\bar{\omega}_\tau = \{t_n = n\tau,\ n = 0, 1, \dots, N_0;\ \tau N_0 = T\} = \omega_\tau \cup T$ with the time step τ, and let us consider in parallelepiped $\bar{G}$ a grid $\bar{\omega}_h = \omega_h \cup \gamma_h$ which is uniform in each direction x_α and where γ_h is the set of all boundary nodes,

$$\bar{\omega}_h = \left\{ x_i = \left(x_1^{(i_1)}, x_2^{(i_2)}, \dots, x_p^{(i_p)} \right),\ x_\alpha^{(i_\alpha)} = i_\alpha h_\alpha, i_\alpha = 1, 2, \dots, N_\alpha - 1,\ h_\alpha N_\alpha = l_\alpha,\ \alpha = 1, 2, \dots, p \right\}.$$

The operators $L_{\alpha\alpha}$ are approximated by corresponding finite difference operators

$$\Lambda_{\alpha\alpha} y = (a_{\alpha\alpha} y_{\bar{x}_\alpha})_{x_\alpha}, \quad a_{\alpha\alpha} = 0,5 \left(k_{\alpha\alpha}^{(-1_\alpha)} + k_{\alpha\alpha} \right), \qquad \alpha = \overline{1,p}, \tag{5.29}$$

where

$$(\pm 1_\alpha) = v\left(x_1^{(i_1)}, \dots, x_{\alpha-1}^{(i_\alpha - 1)}, x_\alpha^{(i_\alpha)} \pm h_\alpha, x_{\alpha+1}^{(i_\alpha+1)}, \dots, x_p^{(i_p)} \right),$$

and the operators $L_{\alpha\beta}$, $\alpha \ne \beta$ are approximated in the following way:

$$\Lambda_{\alpha\beta}^{+} y = 0,5 \left((k_{\alpha\beta} y_{x_\beta})_{x_\alpha} + (k_{\alpha\beta} y_{\bar{x}_\beta})_{\bar{x}_\alpha} \right), \qquad \alpha \ne \beta. \tag{5.30}$$

On the uniform grid $\omega = \omega_h \times \omega_\tau$, one approximates the differential problem (5.25) using the implicit difference scheme

$$y_t = \sum_{\alpha=1}^{p} \Lambda_{\alpha\alpha} \hat{y} + \sum_{\alpha,\beta=1}^{p} \Lambda_{\alpha\beta}^{+} \hat{y} + \varphi, \tag{5.31}$$

$$y(x, 0) = u_0(x), \quad x \in \omega_h, \qquad \hat{y}|_{\gamma_h} = \mu(x, \hat{t}), \quad x \in \gamma_h, \tag{5.32}$$

where $\varphi = \hat{f}$, $y = y(x_i, t_n)$, $x_i \in \omega_h$, $t_n \in \omega_\tau$, $\hat{y} = y^{n+1} = y(x_i, t_{n+1})$, $y_t = (y^{n+1} - y^n)/\tau$. Applying the formulae

$$\Lambda_{\alpha\alpha} y = \frac{\left(a_{\alpha\alpha}^{(+1_\alpha)} \left(y^{(+1_\alpha)} - y \right) - a_{\alpha\alpha} \left(y - y^{(-1_\alpha)} \right) \right)}{h_\alpha^2},$$

$$\Lambda_{\alpha\beta} y = \Big(k_{\alpha\beta}^{(+1_\alpha)} \left(y^{(+1_\beta, +1_\alpha)} - y^{(+1_\alpha)} \right) - k_{\alpha\beta} \left(y^{(+1_\beta)} - y \right) + k_{\alpha\beta} \left(y - y^{(-1_\beta)} \right) - k_{\alpha\beta}^{(-1_\alpha)} \left(y^{(-1_\alpha)} - y^{(-1_\beta, -1_\alpha)} \right) \Big) / (2 h_\alpha h_\beta),$$

$$\sum_{\alpha=1}^{p} \frac{a_{\alpha\alpha}^{(+1_\alpha)}}{h_\alpha^2} y^{(+1_\alpha)} - \sum_{\substack{\alpha,\beta=1 \\ \alpha \ne \beta}}^{p} \frac{\left(k_{\alpha\beta}^{(+1_\alpha)} y^{(+1_\alpha)} + k_{\alpha\beta} y^{(+1_\beta)} \right)}{2 h_\alpha h_\beta} = \sum_{\alpha=1}^{p} D_\alpha y^{(+1_\alpha)},$$

$$\sum_{\alpha=1}^{p} \frac{a_{\alpha\alpha}^{(-1_\alpha)}}{h_\alpha^2} y^{(-1_\alpha)} - \sum_{\substack{\alpha,\beta=1 \\ \alpha \ne \beta}}^{p} \frac{\left(k_{\alpha\beta}^{(-1_\alpha)} y^{(-1_\alpha)} + k_{\alpha\beta} y^{(-1_\beta)} \right)}{2 h_\alpha h_\beta} = \sum_{\alpha=1}^{p} E_\alpha y^{(-1_\alpha)},$$

$$D_\alpha = \frac{a_{\alpha\alpha}^{(+1_\alpha)}}{h_\alpha^2} - \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{p} \frac{k_{\alpha\beta}^{(+1_\alpha)} + k_{\beta\alpha}}{2h_\alpha h_\beta}, \qquad E_\alpha = \frac{a_{\alpha\alpha}^{(-1_\alpha)}}{h_\alpha^2} - \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{p} \frac{k_{\alpha\beta}^{(-1_\alpha)} + k_{\beta\alpha}}{2h_\alpha h_\beta},$$

one can rewrite the differential scheme (5.31) in canonical form (2.6),

$$\begin{aligned} A(x)y(x,t) &= \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{p} \tau\left(B_{\alpha\beta}y^{(+1_\beta,+1_\alpha)} + C_{\alpha\beta}y^{(-1_\beta,-1_\alpha)}\right) \\ &+ \sum_{\alpha=1}^{p} \tau\left(D_\alpha y^{(+1_\alpha)} + E_\alpha y^{(-1_\alpha)}\right) + F(x,t), \qquad (x,t)\in\omega, \end{aligned} \tag{5.33}$$

where

$$B_{\alpha\beta} = \frac{k_{\alpha\beta}^{(+1_\alpha)}}{2h_\alpha h_\beta} \geq 0, \qquad C_{\alpha\beta} = \frac{k_{\alpha\beta}^{(-1_\alpha)}}{2h_\alpha h_\beta} \geq 0,$$

$$A = 1 + \tau\left(\sum_{\alpha=1}^{p} \frac{a_{\alpha\alpha} + a_{\alpha\alpha}^{(+1_\alpha)}}{h_\alpha^2} - \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}} \frac{k_{\alpha\beta}}{h_\alpha h_\beta}\right), \qquad F(x,t) = \tau\varphi + y.$$

THEOREM 3. *Let us suppose that for all $x \in \omega_h$, one has*

$$D_\alpha(x) \geq 0, \quad E_\alpha(x) \geq 0, \qquad \alpha = 1, 2, \ldots, p.$$

Then the solution of the finite difference scheme (5.31),(5.32) is unconditionally stable for each $\tau > 0$ (without any restrictions on τ and h_α, $\alpha = 1, 2, \ldots, p$) in respect to initially conditions, boundary condition, and right side of equation and for any $t_n \in \omega_\tau$, the estimation

$$\|y^{n+1}\|_{\bar{C}} \leq \max\left\{\|y^0\|_{\bar{C}}, \max_{1\leq k\leq n+1} \|y^k\|_{C_\gamma}\right\} + \sum_{k=0}^{n} \tau \|\varphi^k\|_C$$

is valid.

PROOF. As far as the conditions of Lemma 1 are satisfied, using the *a priori* estimation (2.8), one has $\|y^{n+1}\|_{\bar{C}} \leq \max\{\|y^{n+1}\|_{C_\gamma}, \|F\|_C\}$. Let us note that $\|F\|_C \leq \|y^n\|_C + \tau\|\varphi^n\|_C$. Now inserting this estimation in the last inequality, one finds that

$$\begin{aligned} \|y^{n+1}\|_{\bar{C}} &\leq \max\{\|y^{n+1}\|_{C_\gamma}, \|y^n\|_{\bar{C}} + \tau\|\varphi^n\|_C\} \\ &\leq \max\left\{\|y^{n+1}\|_{C_\gamma}, \|y^n\|_{C_\gamma} + \tau\|\varphi^n\|_C, \|y^{n-1}\|_{\bar{C}} + \tau\|\varphi^{n-1}\|_C + \|\varphi^n\|_C\right\} \\ &\leq \cdots \leq \max\left\{\max_{1\leq k\leq n+1} \|y^k\|_{C_\gamma} + \sum_{k=0}^{n} \tau\|\varphi^k\|_C, \|y^0\|_{\bar{C}} + \sum_{k=0}^{n} \tau\|\varphi^k\|_C\right\}. \end{aligned}$$

The required relation follows now from this inequality. The theorem is demonstrated.

NOTE 3. The conditions of Theorem 2 can always be satisfied if the matrix of the coefficients of equation (5.25) $K = \{k_{\alpha\beta}\}_{\alpha,\beta=1}^{p}$ is diagonal dominating with respect to the lines and the columns, that is,

$$k_{\alpha\alpha} \geq \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{p} |k_{\alpha\beta}|, \qquad k_{\alpha\alpha} \geq \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{p} |k_{\beta\alpha}|.$$

In this case, one can set $h_1 = h_2 = \cdots = h_p$.

NOTE 4. In order to construct a monotone difference scheme for coefficients with alternating signs $k_{\alpha\beta}(x)$, $\alpha \neq \beta$, one has to rewrite equation (5.25) in nondivergent form

$$\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{p} \left(k_{\alpha\alpha}(x) \frac{\partial^2 u}{\partial x_\alpha^2} + k_\alpha(x) \frac{\partial u}{\partial x_\alpha} \right) + \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{p} k_{\alpha\beta}(x) \frac{\partial^2 u}{\partial x_\alpha x_\beta} + f(x,t).$$

Now the difference scheme will be monotone if the matrix K is diagonal dominated, and will have the second order of local approximation $O(h_1^2 + h_2^2 + \cdots + h_p^2)$ and will have the following form:

$$y_t = \sum_{\alpha=1}^{p} \left(\frac{k_{\alpha\alpha}}{1+R_\alpha} \hat{y}_{\bar{x}_\alpha x_\alpha} + k_\alpha^+ \hat{y}_{x_\alpha} + k_\alpha^- \hat{y}_{\bar{x}_\alpha} \right) + \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}} \left(k_{\alpha\beta}^+ \Lambda_{\alpha\beta}^+ \hat{y} + k_{\alpha\beta}^- \Lambda_{\alpha\beta}^- \hat{y} \right) + \varphi,$$

where

$$R_\alpha = \frac{0,5|k_\alpha| h_\alpha}{k_{\alpha\alpha}}, \qquad k_\alpha^\pm = 0,5\,(k_\alpha \pm |k_\alpha|)\,,$$

$$k_{\alpha\beta}^\pm = 0,5\,(k_{\alpha\beta} \pm |k_{\alpha\beta}|)\,, \qquad \Lambda_{\alpha\beta}^+ y = y_{\bar{x}_1 \bar{x}_2} + y_{x_1 x_2}, \qquad \Lambda_{\alpha\beta}^- y = y_{\bar{x}_1 x_2} + y_{x_1 \bar{x}_2}.$$

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